SEL4223 Digital Signal Processing

LTI System

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Introduction

- The system is linear and time-invariant.

- Recall from the previous chapter:

  - Linear system – no operation such as power, log, cosine and sine are allowed

  - Time-invariant – variable n must not be accompanied with any mathematical operation except plus and minus.

- Thus, what is left is, the LTI system can be represented as a weighted sum of input and output sequences called difference equation.

LTI System Representation

- There are two methods on how to represent the LTI system.

Difference Equation:	Represents LTI system in terms of its input and output signal
Impulse Response:	Represents LTI system based on the system transfer function

- The transfer function of the LTI system is called impulse response because when the system is input with impulse signal, the output of the system is the transfer function.

Difference Equation

- A sequence of discrete input signal $x[n]$ and output signal $y[n]$, generated recursively based on weight of the input and output sequence.

$$y[n] = -\sum_{k=-\infty}^{\infty} a_k y[n-k] + \sum_{k=-\infty}^{\infty} b_k x[n-k]$$

Output Sequence   Input Sequence

$a_k \rightarrow weight\ of\ output\ sequence$

$b_k \rightarrow weight\ of\ input\ sequence$

Difference Equation (cont.)

- When the LTI system consist of output sequence, the system is said to have a feedback.

- In the LTI system the weight of the input and output sequence will determine the transfer function of the system.

- The name 'difference equation' comes from its continuous version which is called 'differential equation' where differential operation is used to obtain the differential equation from the Laplace transform of the system. For discrete system, Laplace transform is replaced with Z-transform. We will cover the Z-transform in the next chapter.

Example 1
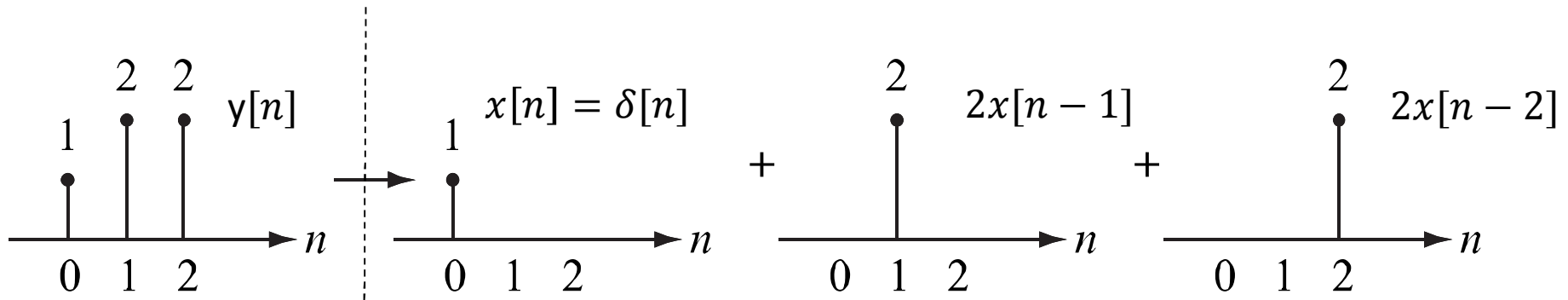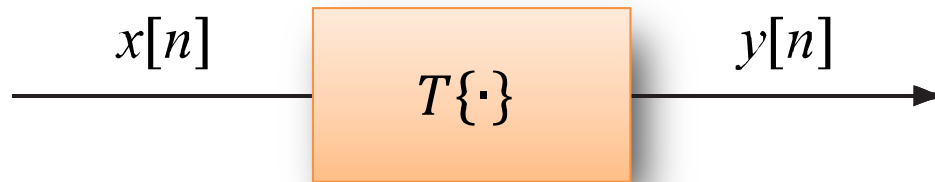

- The difference equation for above system is

$$y[n] = x[n] + 2x[n-1] + 2x[n-2]$$

$$= \sum_{k=0}^{2} b_k x[n-k]$$

- Where the weight, $b_k = [1\ 2\ 2] = \delta[n] + 2\delta[n-1] + 2\delta[n-2]$

Impulse Response

- Impulse response is the transfer function, $T\{\cdot\}$ of an LTI system.

$$x[n] \qquad \boxed{T\{\cdot\}} \qquad y[n]$$

- Derivation:

- $y[n] = T\{x[n]\}$

- $y[n] = T\{\sum_{k=0}^{M} x[k]\delta[n-k]\}$

- Because $x[n]$ is an input, thus, $T\{.\}$ actually refers to $\delta[n-k]$ alone.
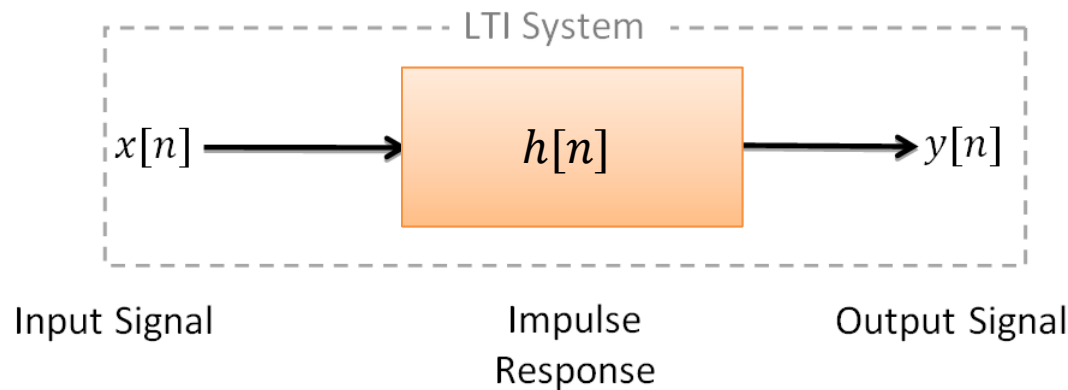
Impulse Response (cont.)

- $y[n] = \sum_{k=0}^{M} x[k] T\{\delta[n-k]\} = \sum_{k=0}^{M} x[k] h[n-k]$

- Where $T\{\delta[n-k]\} = h[n-k]$. Because input to $h[n]$ is $\delta[n]$ (impulse signal), $h[n]$ is called impulse response and $h[n] = b_k$

- Based on the difference equation definition, what does it mean by the weight sum of input sequence in $y[n]$ below?

- $y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$

- From equation above, it can be proved that

- $y[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$

Impulse Response (cont.)

- Here, it shows that the summation range depends on the length of $h[k]$. Let's say that $h[k]$ has nonzero values from $k = 0$ to $k = 2$. Thus,

- $y[n] = \sum_{k=0}^{2} h[k]x[n-k]$

- $y[n] = h[0]x[n] + h[1]x[n-1] + h[2]x[n-2]$

- Whereby $x[\ ]$ is the input sequence and $h[\ ]$ is the weight value for input signal $x[n]$.

Impulse Response (cont.)

- In LTI system, instead of the input and output signals, the transfer function $T\{.\}$ is also represented as discrete signal called impulse response, $h[n]$ which act as the weight to the input and output sequences.

Input Signal   Impulse Response   Output Signal

- Thus, designing the discrete LTI system (the transfer function) is actually a process of finding or fine tuning the weight values.

Example 2

- Find the impulse response of the difference equation below

$$y[n] = x[n] + 2x[n-1] - 3x[n-3]$$

Solution:

- Impulse response is the weight of the input sequence. Thus

$$h[n] = [1, 2, 0, -3] = \delta[n] + 2\delta[n-1] - 3\delta[n-3]$$

*Note: Definition of impulse response as the weight of input sequence is only true if there is **no past and future output**. We will look at a broader definition of the impulse response in future topic.

Types of Impulse Response

- There are 2 types of impulse response:

    – Finite Impulse Response (FIR)

    – Infinite Impulse Response (IIR)

- The finite and infinite term are referring to the size or length of the impulse response.

FIR System

- Difference equation for FIR system is shown below where there is no output sequence.

$$y[n] = \sum_{k=N}^{M} b_k x[n-k]$$

- $M - N$ is a finite number where $M \geq N$. Both M and N can be either positive or negative integer number.

- b_k is the weight to the input sequence, which is also the impulse response for the FIR system. Thus, $b_k = h[k]$

Example 3

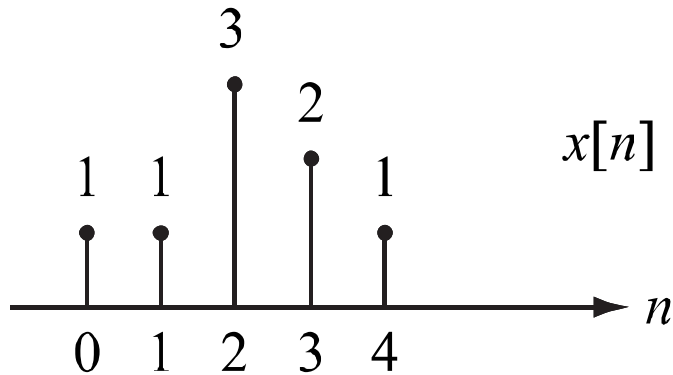- If the impulse response of a system is $h[n] = [\frac{1}{2}, \frac{1}{2}]$, find $y[n]$

Solution

- The output of the FIR system can be obtain from the system difference equation where

$$y[n] = \sum_{k=0}^{1} h[k]x[n-k]$$

$$= h[0]x[n] + h[1]x[n-1]$$

$$= \frac{1}{2}x[n] + \frac{1}{2}x[n-1]$$

- The following is an example when $x[n]$ is known

Example 3 (cont.)
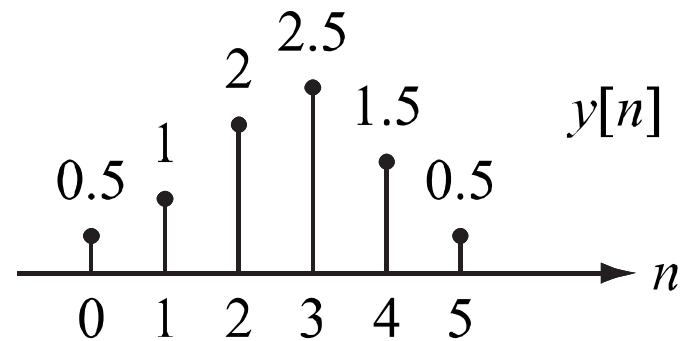
$$y[0] = \frac{1}{2}(1) + \frac{1}{2}(0) = 0.5$$

$$y[3] = 2.5$$

$$y[1] = \frac{1}{2}(1) + \frac{1}{2}(1) = 1$$

$$y[4] = 1.5$$
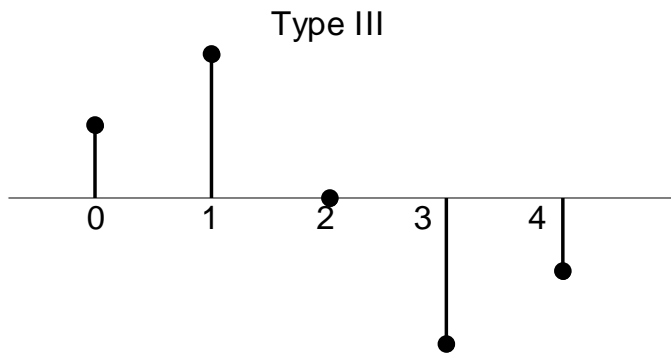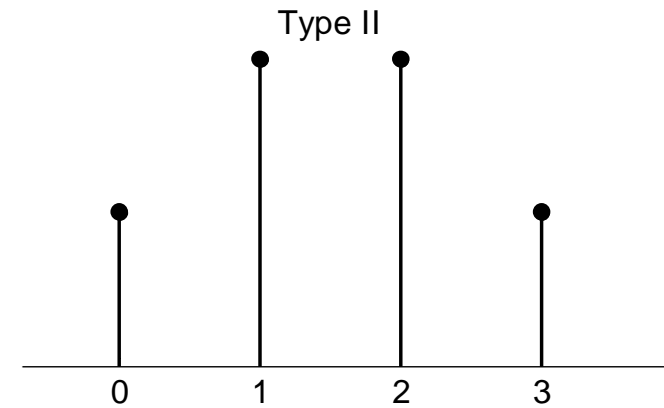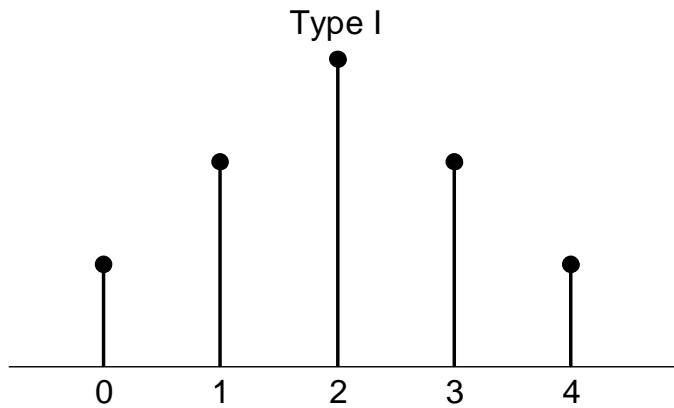
$$y[5] = 0.5$$

$$y[2] = \frac{1}{2}(3) + \frac{1}{2}(1) = 2$$

Special FIR System

- For FIR system, there are special type of the system as below where its impulse response is symmetry and nonzero samples only exist at $n \geq 0$

Type	Impulse Response	Length $(M + 1)$
I	$h[n] = h[M - n]$	Odd
II	$h[n] = h[M - n]$	even
III	$h[n] = -h[M - n]$	odd
IV	$h[n] = -h[M - n]$	even

Special FIR System (cont.)
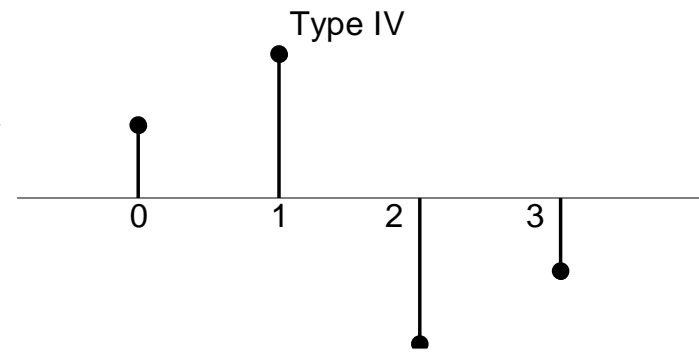
Convolution

- The mathematical operation of weight sum of the input sequence shown so far in the FIR system is a special mathematical operation called convolution. From the FIR equation, it can be proved that convolution is represented as below

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- There are 4 operation in the convolution:

  I.   Summation

  II. Weight multiplication

  III. Delay

  IV. Folding

- Convolution can also be denoted as $y[n] = x[n] * h[n]$

Convolution Steps

1. Rewrite $x[n]$ and $h[n]$ as $x[k]$ and $h[k]$

2. Do folding on $h[k]$ to obtain $h[-k]$

3. Put $x[k]$ and $h[-k]$ on the same axis and delay $h[-k]$ by $n = n_s$ to obtain $h[n_s - k]$:
   ($n_s =$ the lowest n in $x[n]$ + the lowest n value in $h[n]$)

4. Multiply $x[n]$ with $h[n-k]$ and sum up the result

5. Delay $h[n-k]$ by 1 sample to the right and repeat step (4)

6. Repeat step (5) until $n = n_e$:
   ($n_e =$ highest n value in $x[n]$ + highest n value in $h[n]$)

Example 4

- Find $y[n]$, the output of a system where input signal $x[n]$ and its impulse response $h[n]$ are as follow
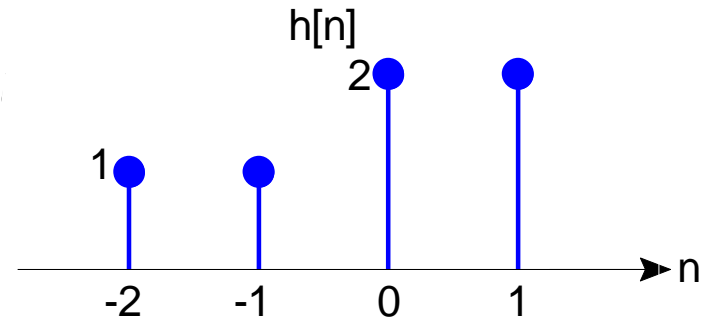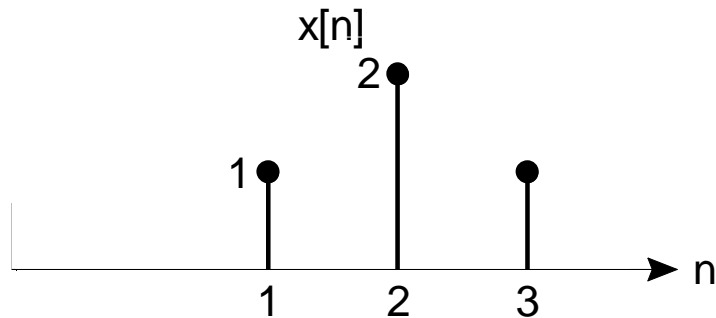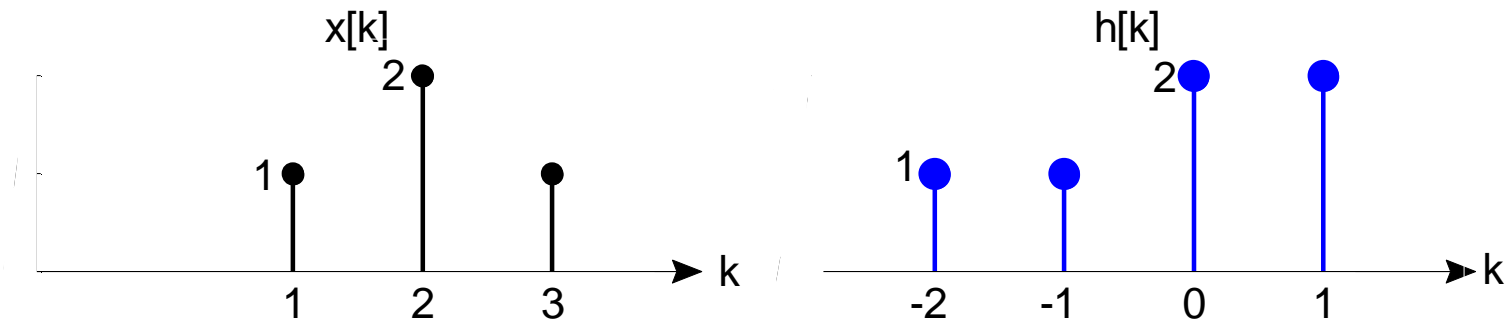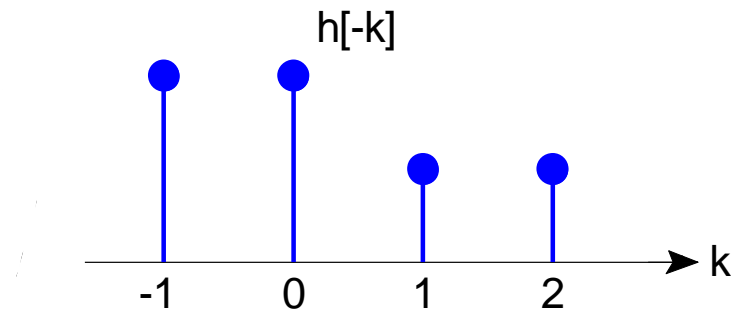

Solution

- It can be done by performing convolution between $x[n]$ and $h[n]$

Example 4 (cont.)

1. Rewrite $x[n]$ and $h[n]$ as $x[k]$ and $h[k]$
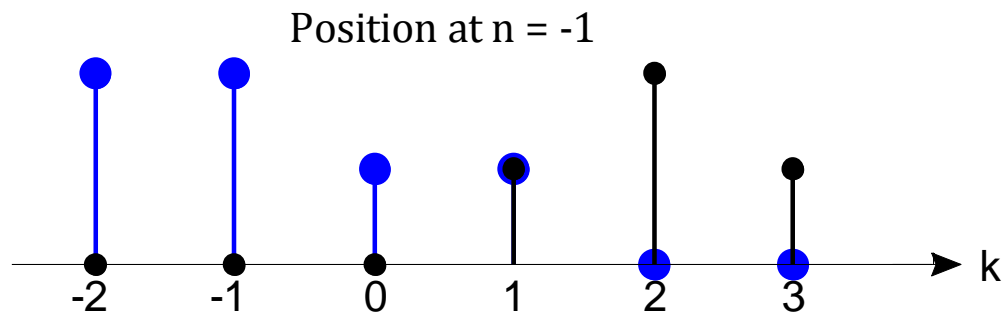

2. Obtain $h[-k]$

Example 4 (cont.)

3. Put $x[k]$ and $h[-k]$ at similar x-axis and delay $h[-k]$ by n_s

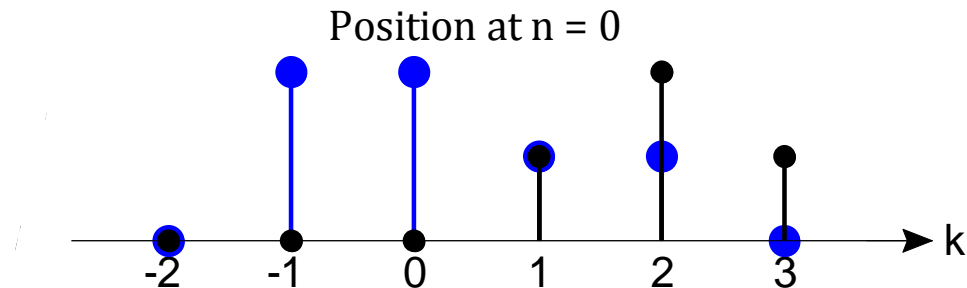 n_s = the lowest n in $x[n]$ + the lowest n value in $h[n]$

   $\quad = -2 + 1 = -1.$

   In other words, move $h[-k]$ until the most right sample of the $h[-k]$ is on top of the most left sample of the $x[k]$

Position at n = -1

Example 4 (cont.)

4. Multiply $x[k]$ and $h[-1-k]$ and sum up the values. At this position, $n = -1$. Thus, the operation is referred to $y[-1]$

$$y[-1] = x[1]h[-1-1] = x[1]h[-2] = 1(1) = 1$$
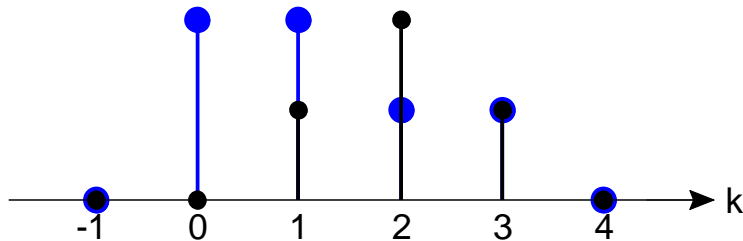
5. Delay $h[n-k]$ by 1 sample and repeat step (4)

Position at n = 0

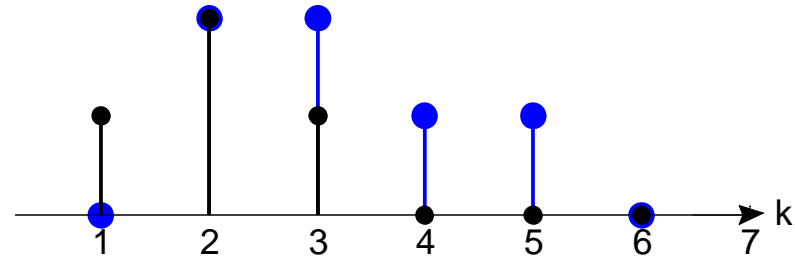Now, $n = 0$. Thus, $y[0] = 1(1) + 1(2) = 3$

Example 4 (cont.)

6. Repeat step (5) until $n = n_e$

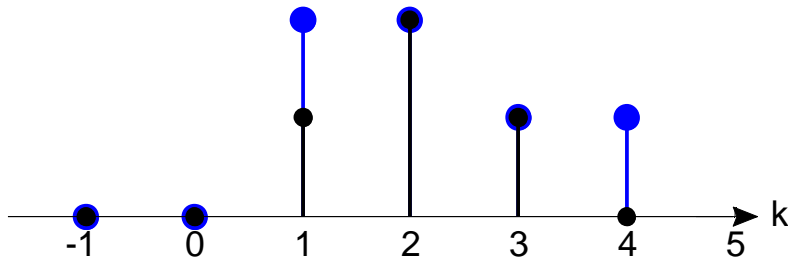$n_e = $ highest n value in $x[n] + $ highest n value in $h[n] = 1 + 3 = 4$

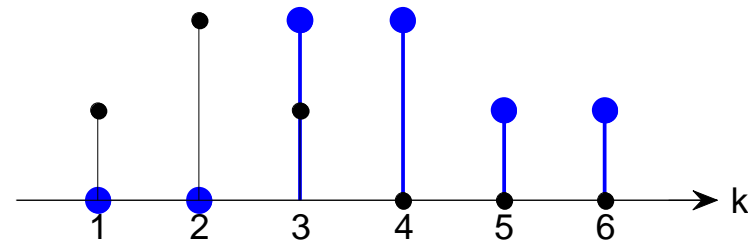For $n = 1, y[1] = 2(1) + 1(2) + 1(1) = 5$

For $n = 3, \ y[3] = 2(2) + 2(1) = 6$
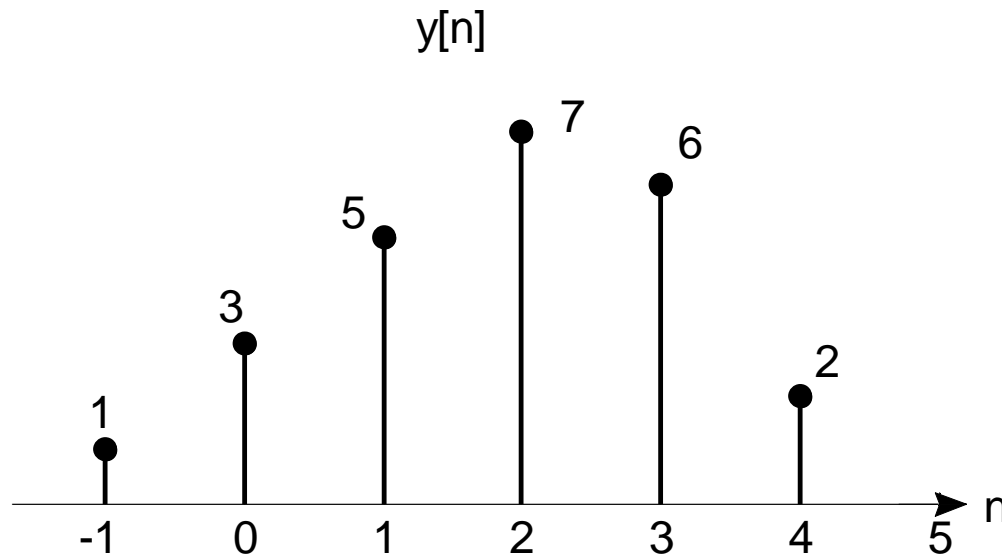

For $n = 2, \ y[2] = 2(1) + 2(2) + 1(1) = 7$

For $n = 4, \ y[4] = 2(1) = 2$

Example 4 (cont.)

- Finally, we can sketch $y[n]$ as below

Example 5

Find $y[n] = x[n] * h[n]$

$h[n]$ | 1 | 1 | 2 |

$x[n]$ | 5 | 6 |

At $n = 0, y[n] = 5$

$h[-k]$ | 2 | 1 | 1 |

$x[k]$ | 5 | 6 |

-2   -1   0   1   2   3   4

At $n = 1, y[n] = 11$

$h[1-k]$ | 2 | 1 | 1 |

$x[k]$ | 5 | 6 |

-2   -1   0   1   2   3   4

At $n = 2, y[n] = 16$

$h[2-k]$ | 2 | 1 | 1 |

$x[k]$ | 5 | 6 |

-2   -1   0   1   2   3   4

At $n = 3, y[n] = 12$

$h[3-k]$ | 2 | 1 | 1 |

$x[k]$ | 5 | 6 |

-2   -1   0   1   2   3   4

Quiz 1

For $h[n] = u[n] - u[n-4]$, find $y[n]$ if $x[n]$ is given as follows. Also, plot $y[n]$.

1. $x[n] = h[n]$

2. $x[n] = \delta[n+2] + \delta[n-2]$

3. $x[n] = \alpha^n u[-n-3] - \alpha^n u[-n-4]$

4. $x[n] = \sum_{k=0}^{5} \delta[n-6k]$

5. $x[n] = \alpha^{n-\beta} (u[n] - u[n-1])$

6. $x[n] = 2h[n]$

7. $x[n] = u[n]$

8. $x[n] = \sin(0.5\pi n)$

9. $x[n] = \cos(0.5\pi n) + \alpha$

10. $x[n] = n\delta[n] + \delta[n-1] + \delta[n-2]$

IIR System

- IIR – Infinite Impulse Response – Length of $h[n]$ is infinite.

- In general, the difference equation of IIR system is shown below where output sequence is added to the equation compare to FIR system

$$y[n] = -\sum_{k=-\infty}^{\infty} a_k y[n-k] + \sum_{k=-\infty}^{\infty} b_k x[n-k]$$

Difference Equation Solution

- For IIR systems, convolution will cause an endless computation as the length is infinite (remember that step (6) of the convolution process is to repeat the computation until $n = n_e$. For IIR system, $n_e = \infty$).

- Below are the methods for obtaining the difference equation

System	Method
FIR	- Convolution - Z-transform
IIR	- Z-transform - Initial value assumption (not covered)

- Z-transform will be discussed in the next chapter

Quiz 2

Sketch and determine whether the following $h[n]$ corresponds to an IIR or FIR system.

1. $h[n] = u[n]$

2. $h[n] = \delta[n]$

3. $h[n] = [1\ 3\ 5\ 6\ \infty]$

4. $h[n] = \alpha^n$

5. $h[n] = \alpha^n u[n]$

6. $h[n] = u[n] - u[n-5]$

7. $h[n] = u[n] - \delta[n-100]$

8. $h[n] = \delta[n-2] + \delta[n+2] + \delta[n]$

9. $h[n] = \alpha^n u[-n]$

10. $h[n] = \sum_{n=-10}^{\infty} nx[n]$

11. $h[n] = \sin(n)\,u[n]$

12. $h[n] = \sum_{k=0}^{n} \sin(0.5\pi k)$

13. $h[n] = \sum_{k=0}^{\infty} \sin(0.5\pi k)$

14. $h[n] = e^{jn} u[n]$

15. $h[n] = e^{jn} u[n] - e^{jn} u[n-5]$

Properties of an LTI System

Commutative

$$y[n] = x[n] * h[n] = h[n] * x[n]$$

Associative

$$y[n] = (x[n] * h_1[n]) * h_2[n] = x[n] * (h_1[n] * h_2[n])$$

Distributive

$$y[n] = x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$

Properties of an LTI System (cont.)

Stability

- In previous chapter, stability of discrete-time system is achieved when BIBO condition applied.

- Because LTI system can be represented with impulse response $h[n]$, instead of BIBO, stability of an LTI system can also be described with its impulse response. Below is the rule.

$$\sum_{n=-\infty}^{\infty} |h[n]| \leq \beta_n \neq \infty$$

Properties of an LTI System (cont.)

Causality

- For LTI system, the difference equation without past and future output is

- $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$ or $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$

- For a causal system, no future inputs are involved. Thus, the range of k should start with 0 where $y[n] = \sum_{k=0}^{\infty} h[k]x[n-k]$ or

$$h[n] = 0 \text{ for } n < 0$$

Properties of an LTI System (cont.)

Impulse

$$x[n] * \delta[n] = x[n]$$

$$x[n] * \delta[n - n_d] = x[n - n_d]$$