SEL4223 Digital Signal Processing

Discrete-Time Fourier Transform Properties

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DTFT Properties

Property	$h[n]$	$H(\omega)$
Linear	$ah_1[n] + bh_{2[n]}$	$aH_1(\omega) + bH_2(\omega)$
Time-shifting	$h[n - n_d]$	$e^{-j\omega n_d}H(\omega)$
Frequency-shifting	$e^{j\omega_0 n}h[n]$	$H(\omega - \omega_0)$
Time-reversal	$h[-n]$	$H(-\omega)$
Differentiation	$nh[n]$	$j\left(\frac{dH(\omega)}{d\omega}\right)$
Convolution	$h_1[n] * h_2[n]$	$H_1(\omega)H_2(\omega)$
Multiplication	$h_1[n]h_2[n]$	$H_1(\omega) * H_2(\omega)$

Symmetry Properties

- By substituting $e^{-j\omega n} = \cos(\omega n) - jsin(\omega n)$ into FT formulation, the real and imaginary part of the transform can be separated.

$$H(\omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} [h[n]\cos(\omega n) - jh[n]\sin(\omega n)]$$

$$H_R(\omega) = \sum_{n=-\infty}^{\infty} h[n]\cos(\omega n)$$

$$H_I(\omega) = -\sum_{n=-\infty}^{\infty} h[n]\sin(\omega n)$$

Symmetry Properties (cont.)

- Since $\cos(\omega n) = \cos(-\omega n)$ and $\sin(\omega n) = -\sin(-\omega n)$, thus, it follows that

$$H_R(\omega) = H_R(-\omega), \quad (even)$$
$$H_I(\omega) = -H_I(-\omega), \quad (odd)$$

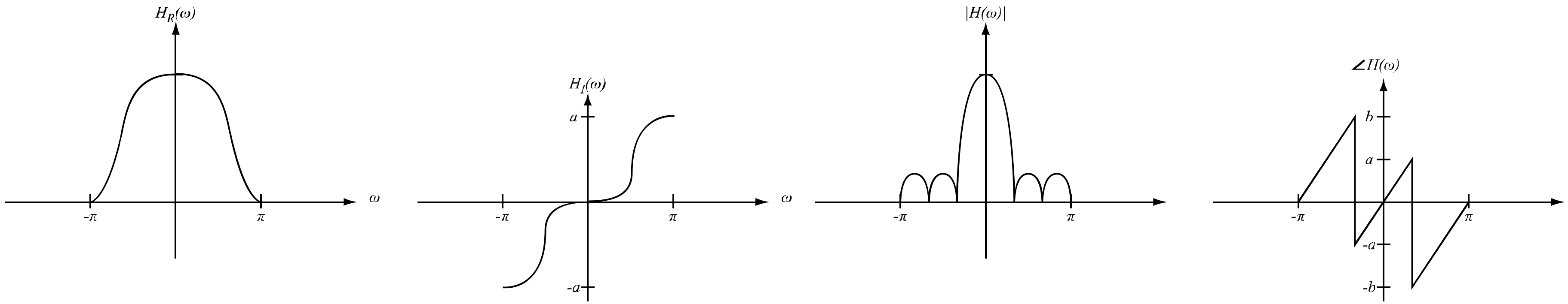
or

$$H^*(\omega) = H(-\omega)$$

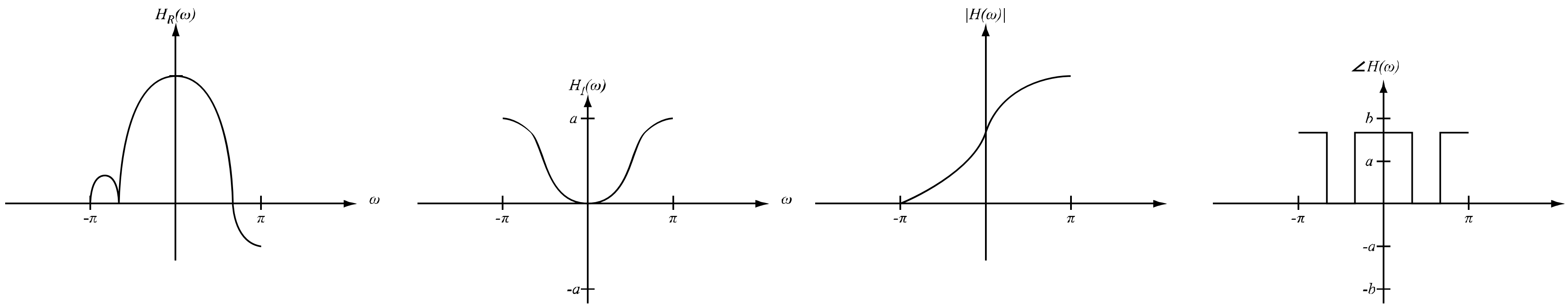
- The magnitude and phase spectra also follows that

$$|H(\omega)| = |H(-\omega)|, \qquad (even)$$
$$\angle H(\omega) = -\angle H(-\omega), \qquad (odd)$$

Symmetry Properties (cont.)



Correct shapes of frequency response



False shapes of frequency response

Symmetry Properties (cont.)

Time domain	Frequency Domain
Any real signal $\mathbf{x[n]}$	$X(\omega) = X^*(-\omega)$ $X_R(\omega) = X_R(-\omega)$ $X_I(\omega) = -X_I(-\omega)$ $\lvert X(\omega)\rvert = \lvert X(-\omega)\rvert$ $\angle X(\omega) = -\angle X(-\omega)$
Real and even signal $\boldsymbol{x[n] = x[-n]}$	$X(\omega) = X_R(\omega) = x[0] + 2\sum_{n=1}^{\infty} x[n]\cos(\omega n)$
Real and odd signal $\boldsymbol{x[n] = -x[-n]}$	$X(\omega) = X_I(\omega) = -j2\sum_{n=1}^{\infty} x[n]\sin(\omega n)$

Example 1

- Find $H(\omega)$, $|H(\omega)|$, $\angle H(\omega)$ and $|H(\omega)|_{dB}$ for:

  $h[n] = 2\delta[n+2] + 2\delta[n-2]$

Solution:

- $h[n]$ is real and even, thus:
- $H(\omega) = h[0] + 2\sum_{n=1}^{\infty} h[n]\cos(\omega n)$

  $= 0 + 2\sum_{n=2}^{2} h[n]\cos(\omega n)$

  $= 4\cos(2\omega)$

Example 1 (cont.)

- $|H(\omega)| = |4\cos(2\omega)|$

- $\angle H(\omega) = \tan^{-1}\left(\frac{0}{4\cos(2\omega)}\right) = \begin{cases} 0 \; if \; H(\omega) \geq 0 \\ \pi \; if \; H(\omega) < 0 \\ -\pi \; if \; H(-\omega) < 0 \end{cases}$

- $|H(\omega)|_{dB} = 20\log_{10}|4\cos(2\omega)|$

Example 2

- Find $H(\omega)$, $|H(\omega)|$, $\angle H(\omega)$ and $|H(\omega)|_{dB}$ for:

  $h[n] = 2\delta[n+2] - 2\delta[n-2]$

Solution:

- $h[n]$ is real and odd, thus:
- $H(\omega) = -j2\sum_{n=1}^{\infty} h[n]\sin(\omega n)$

  $= -j2\sum_{n=2}^{2} h[n]\sin(\omega n)$

  $= -j4\sin(2\omega)$

Example 2 (cont.)

- $|H(\omega)| = |4\sin(2\omega)|$
- $\angle H(\omega) = \tan^{-1}\left(\frac{-4\sin(2\omega)}{0}\right)$

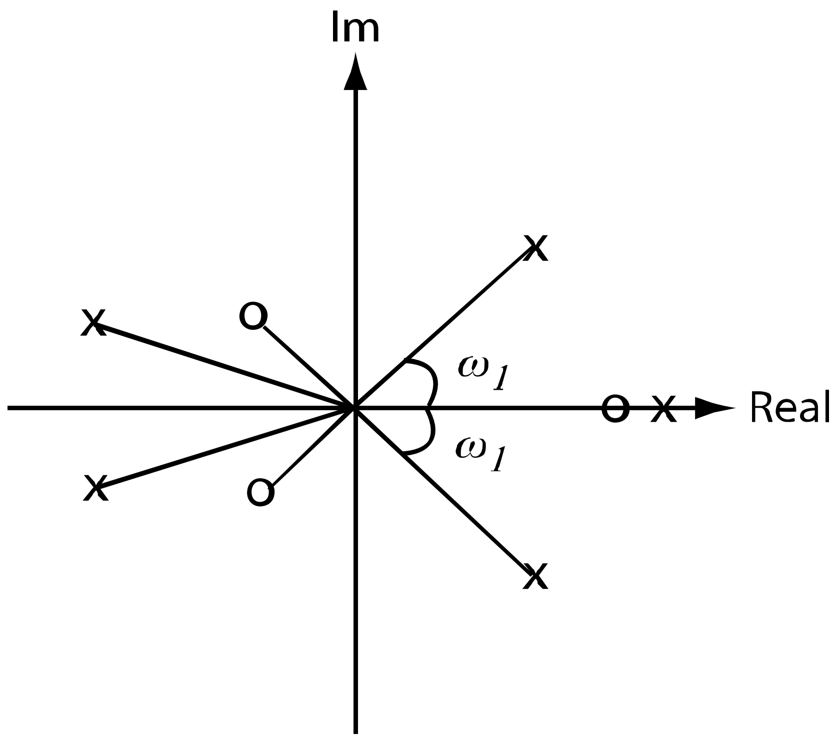
$$= \begin{cases} -\pi/2 & if\ \omega > 0 \\ \pi/2 & if\ \omega < 0 \end{cases}$$

- $|H(\omega)|_{dB} = 20\log_{10}|4\sin(2\omega)|$

Frequency Response related to Poles and Zeros

- Because frequency response for LTI system is symmetry, poles and zeros will exist as a conjugate pair on the z-plane.
- Thus, when there is a pole/zero at $z = re^{j\omega}$, there will also be a pole/zero at $z = re^{-j\omega}$.
- When the poles/zeros are real $(\omega = 0 \; or \; \pi, z \rightarrow r)$, there will be no pair

Poles and Zeros Conjugate

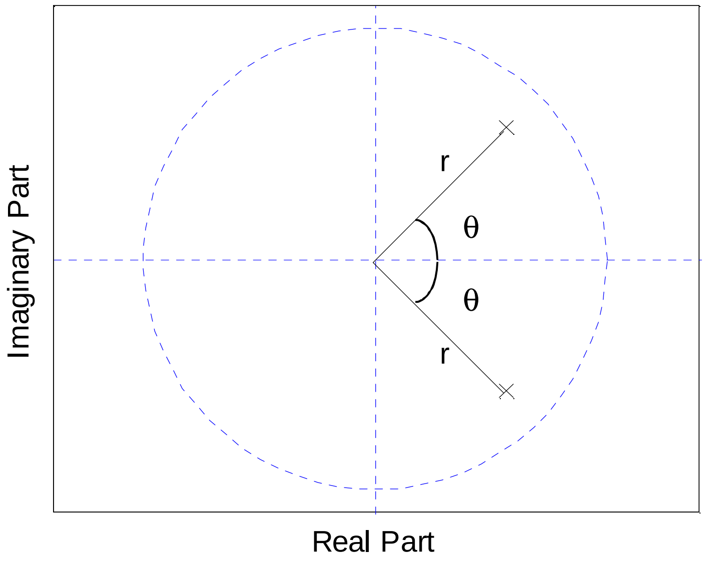


Freq. Response for a Pair of Poles

- System function when only a pair of poles exist is

- $$H(z) = \frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})}$$

$$= \frac{A_1}{1 - re^{j\theta}z^{-1}} + \frac{A_2}{1 - re^{-j\theta}z^{-1}}$$

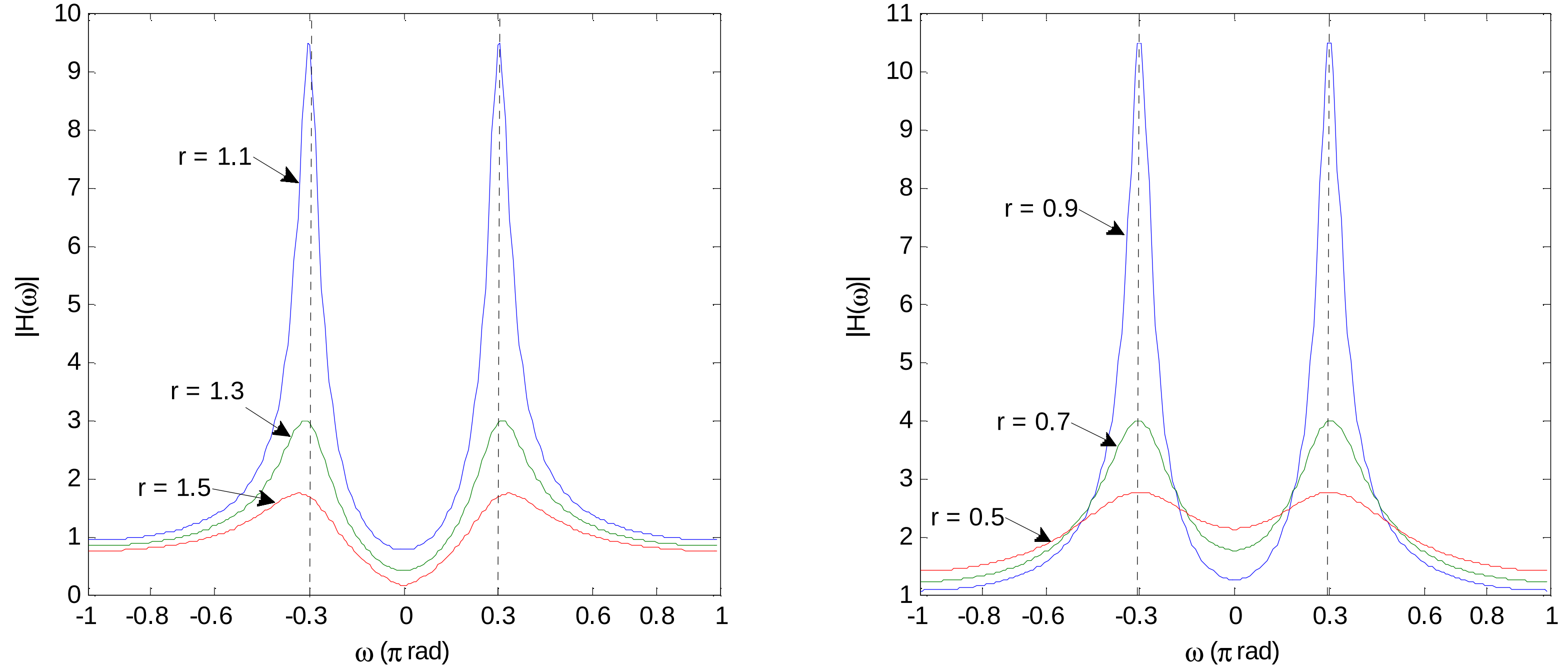


Freq. Response for a Pair of Poles

- As $z = e^{j\omega}$
- $H(\omega) = \frac{A_1}{1-re^{j\theta}e^{-j\omega}} + \frac{A_2}{1-re^{-j\theta}e^{-j\omega}}$

  $= \frac{A_1}{1-re^{-j(\omega-\theta)}} + \frac{A_2}{1-re^{-j(\omega+\theta)}}$
- Based on the equation, $H(\omega)$ is maximum when $\omega = \theta$ and $\omega = -\theta$, where $e^{-j(\omega-\theta)}$ and $e^{-j(\omega+\theta)}$ becomes 1.
- For both causal and stable system, $0 < r < 1$. Within this range, $\max[H(\omega)]$ will increase when r increases as shown as follow.

Freq. Response for a Pair of Poles



Frequency response for a pair of poles with $\theta = 0.3\pi$ across various values of r

Example 3

Plot $|H(\omega)|$ for $h[n] = 2\cos(\omega_o n)\, u[n]$

Solution:

- $H(\omega) = \sum_{n=0}^{\infty}\left(e^{j\omega_0 n} + e^{-j\omega_0 n}\right)e^{-j\omega n}$

$$= \sum_{n=0}^{\infty} e^{j\omega_0 n} e^{-j\omega n} + \sum_{n=0}^{\infty} e^{-j\omega_0 n} e^{-j\omega n}$$

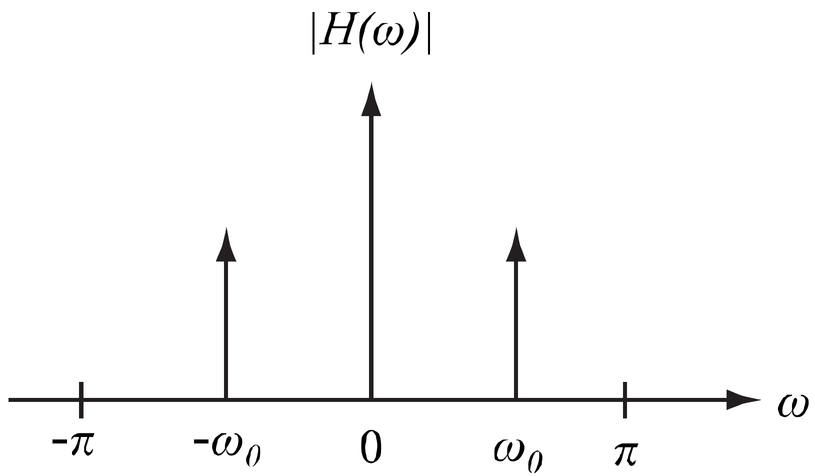
$$= \sum_{n=0}^{\infty} e^{-j(\omega-\omega_0)n} + \sum_{n=0}^{\infty} e^{-j(\omega+\omega_0)n}$$

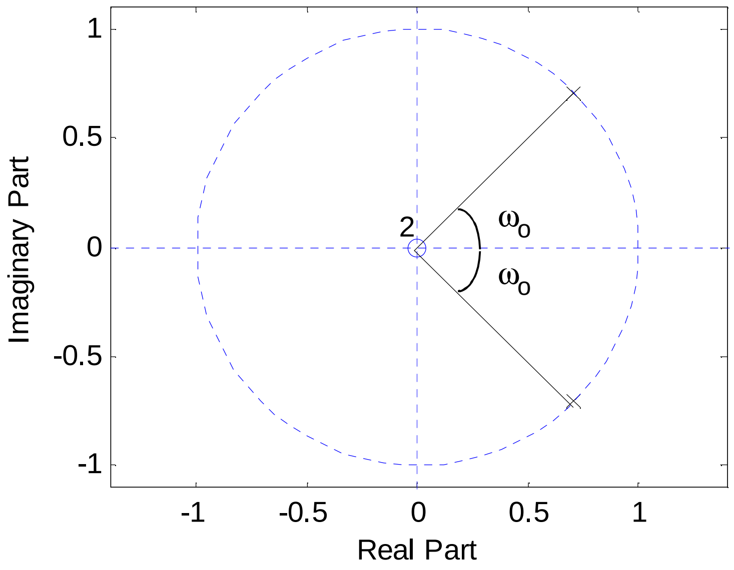
$$= \frac{1}{1-e^{-j(\omega-\omega_0)}} + \frac{1}{1-e^{-j(\omega+\omega_0)}}$$

Example 3 (cont.)

- When $\omega = \omega_0$ or $-\omega_0$, $H(\omega) \to \infty$. Thus, use Hospital's rule. Then, we obtain

$$|H(\omega)| = \delta[\omega - \omega_0] + \delta[\omega + \omega_0].$$





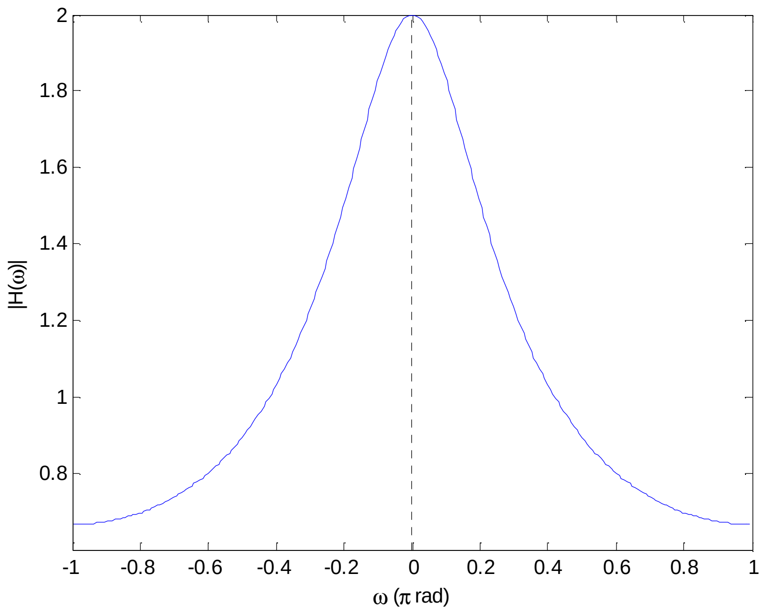
Example 4

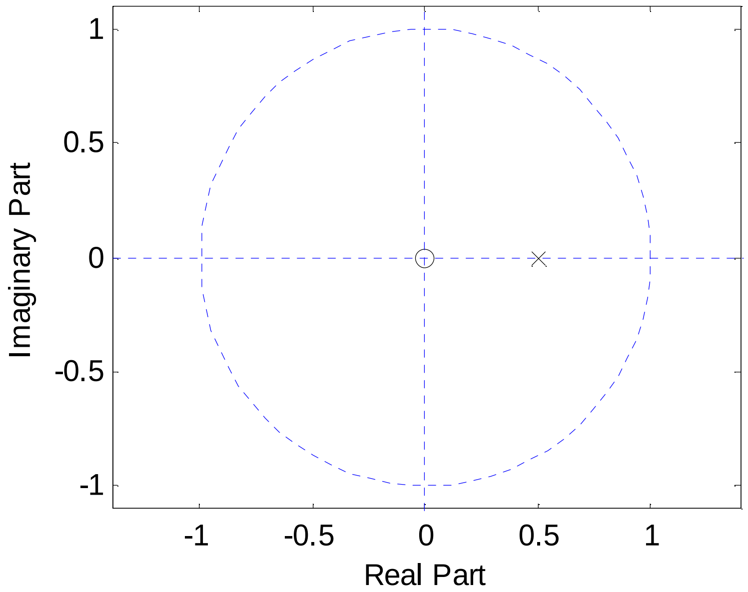
Plot $|H(\omega)|$ for $h[n] = 0.5^n u[n]$

Solution:

- $H(z) = \frac{1}{1-0.5z^{-1}}$
- $H(\omega) = \frac{1}{1-0.5e^{-j\omega}}$
- It has 1 pole at $z = 0.5$ and 1 zero at $z = 0$
- $H(w)$ is maximum when $\omega = 0$
- $H(\omega)|_{max} = \frac{1}{1-0.5} = 2$

Example 4 (cont.)





Freq. Response for a Pair of Zeros

- System function when only a pair of zeros exist is

$$H(z) = \left(1 - re^{j\theta}z^{-1}\right)\left(1 - re^{-j\theta}z^{-1}\right)$$

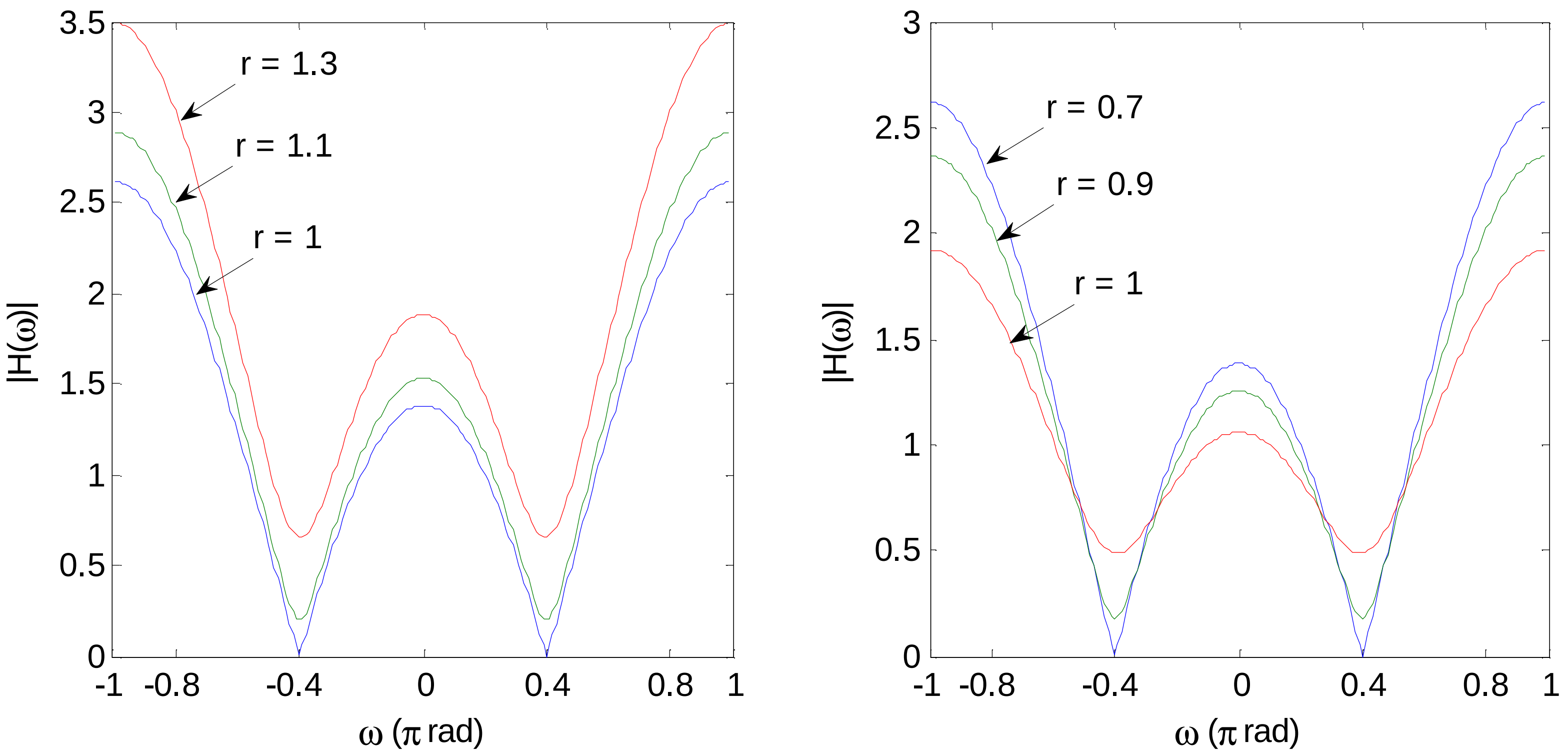
$$H(\omega) = \left(1 - re^{j\theta}e^{-j\omega}\right)\left(1 - re^{-j\theta}e^{-j\omega}\right)$$

$$= \left(1 - re^{-j(\omega-\theta)}\right)\left(1 - re^{-j(\omega+\theta)}\right)$$

- $H(\omega)$ is minimum when:

$\omega = \theta$ and when $\omega = -\theta$

Freq. Response for a Pair of Zeros



Frequency response for a pair of zeros with $\theta = 0.4\pi$ across various values of $\theta = 0.4\pi$

Example 5

- $h[n] = [1\ 0.5\ 0.25]$
  $\uparrow$

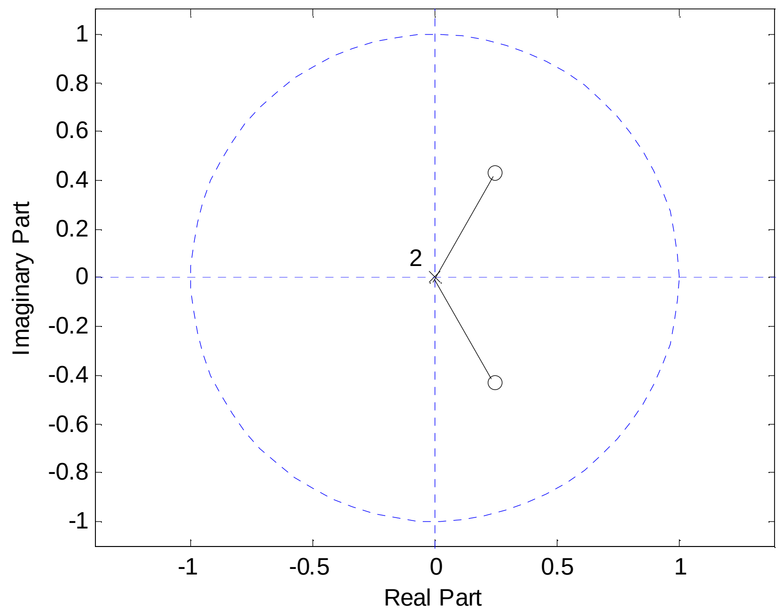
Solution:

- $h[n] = \delta[n] + 0.5\delta[n-1 + 0.25\delta[n-2]$
- $H(z) = 1 + 0.5z^{-1} + 0.25z^{-2}$

  $= (1 - az^{-1})(1 - bz^{-1})$

- $a = 0.25 + j0.433 = 0.5j^{60^0}$, $\quad b = 0.25 - j0.433 = 0.5e^{-j60^0}$
- $H(z) = \left(1 - 0.5e^{j60^0}\right)\left(1 - 0.5e^{-j60^0}\right)$

Example 5 (cont.)



- $H(\omega) = \left(1 - 0.5e^{j60^0}e^{-j\omega}\right)\left(1 - 0.5e^{-j60^0}e^{-j\omega}\right)$

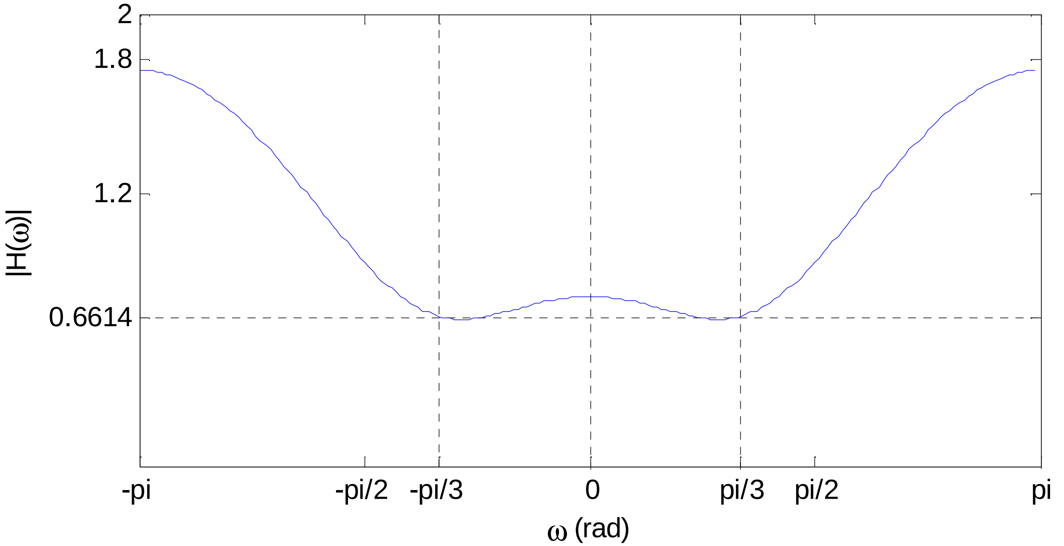
$$= \left(1 - 0.5e^{-j(\omega - 60^0)}\right)\left(1 - 0.5e^{-j(\omega + 60^0)}\right)$$

Example 5 (cont.)

- $H(\omega)$ is minimum when $\omega = 60^0/\frac{\pi}{3}$ or $\omega = -60^0/-\frac{\pi}{3}$
- $H(60^0) = H(\omega)|_{min} = (1 - 0.5)(1.2500 + j0.4330)$

$$= 0.6250 + j0.2165$$

- $|H(60^o)| = 0.6614$



Example 6

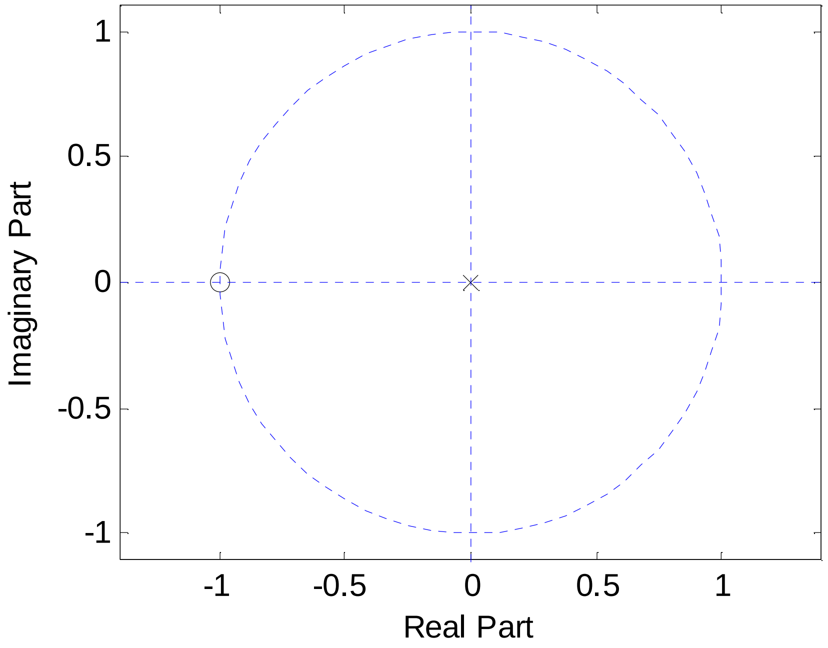
- $H[n] = u[n] - u[n-2]$

Solution:

- $h[n] = u[n] - u[n-2]$

  $= \delta[n] + \delta[n-1]$

- $H(z) = 1 + z^{-1}$
- $H(\omega) = 1 + e^{-j\omega}$
- It has 1 zero at $z = -1$ and 1 pole at $z = 0$



Example 6 (cont.)

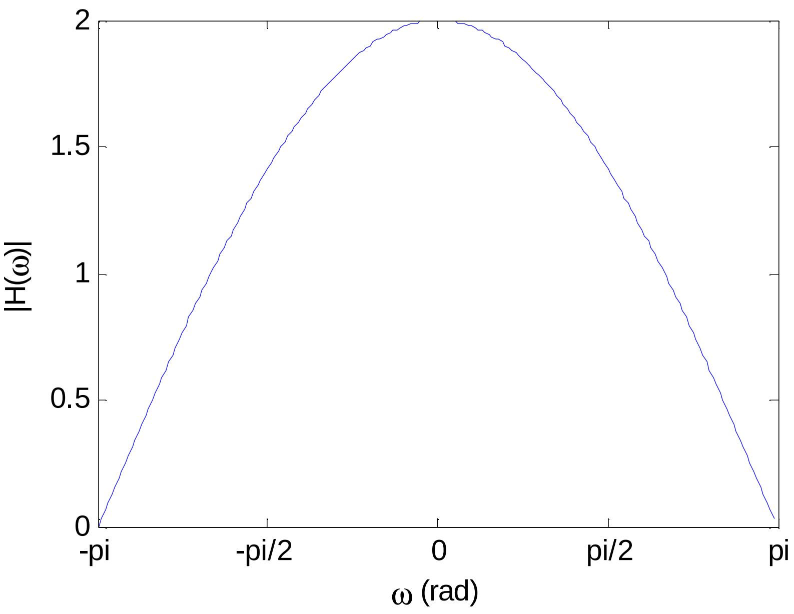
- $H(\omega)$ is minimum at $\omega = \pi$
- $H(\omega)|_{min} = H(\pi) = 1 + e^{-j\pi}$

$$= 1 + \cos(\pi)$$

$$= 1 + (-1) = 0$$

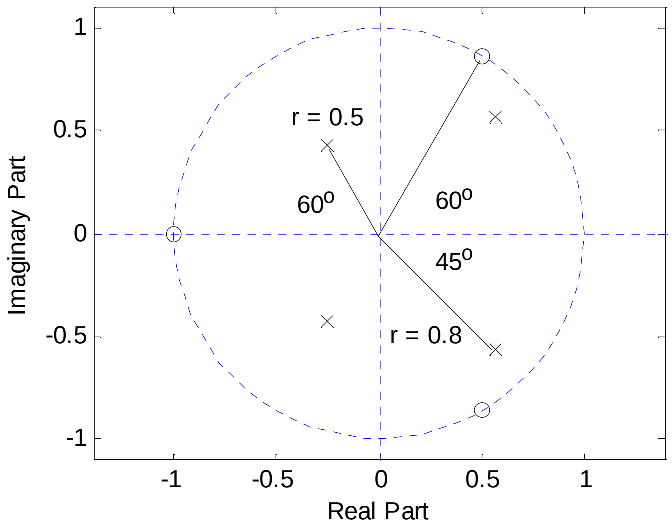
- $H(\omega)$ is maximum at $\omega = 0$
- $H(\omega)|_{max} = H(0) = 1 + e^{0}$

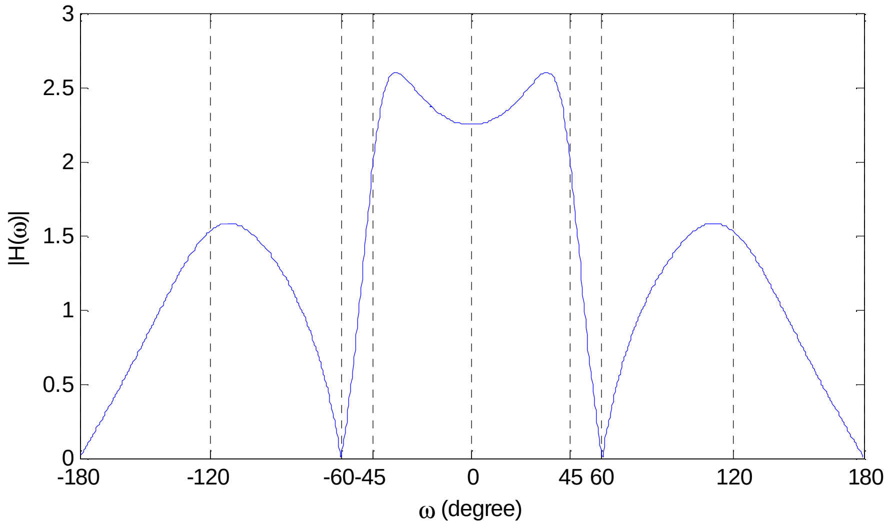
$$= 1 + 1 = 2$$



Freq. Response for Multiple Pairs of Poles and Zeros

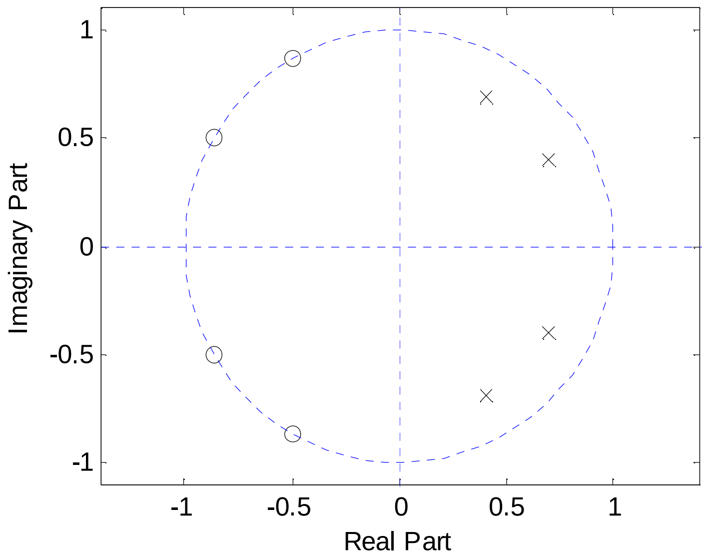
- At each pole, $|H(\omega)|$ goes towards infinity, at each zero, $|H(\omega)|$ goes towards zero

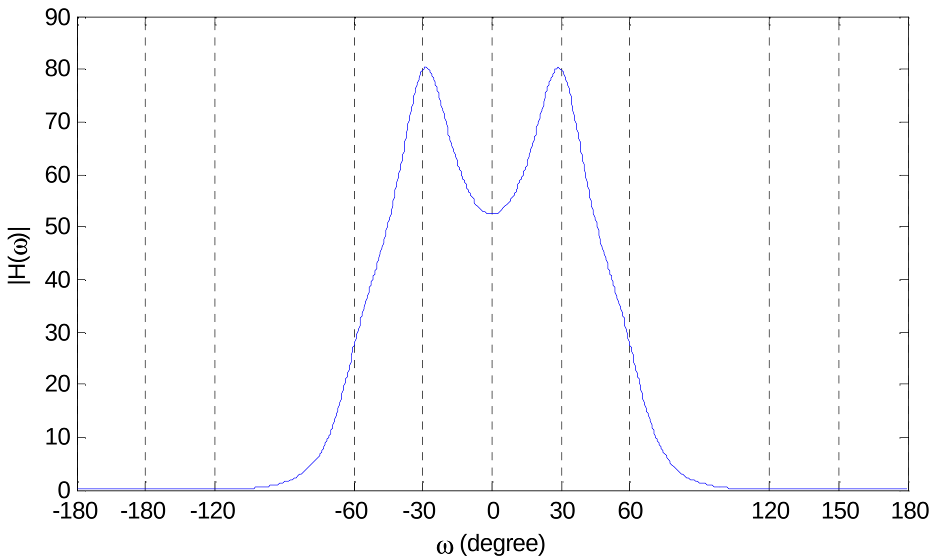




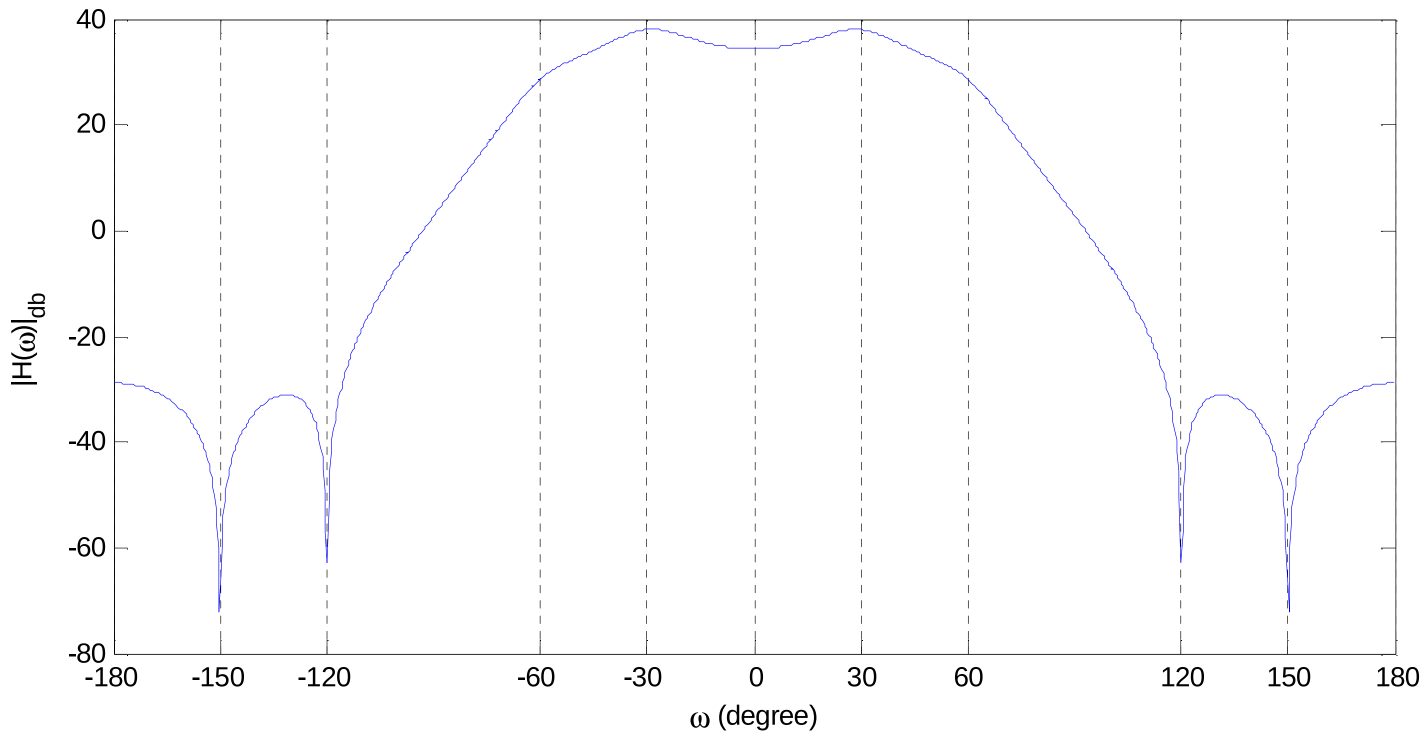
Example 7

- Poles: $0.8e^{j30^o},\ 0.8e^{-j30^o},\ 0.8e^{j60^o},\ 0.8e^{-j60^o}$
- Zeros: $e^{j120^o},\ e^{-j120^o},\ e^{j150^o},\ e^{-j150^o}$





Example 7 (cont.)



Magnitude in db

References

1) John G. Proakis, Dimitris K Manolakis, "Digital Signal Processing: Principle, Algorithm and Applications", Prentice-Hall, 4th edition (2006).

2) Sanjit K. Mitra, "Digital Signal Processing-A Computer Based Approach", McGraw-Hill Companies, 3rd edition (2005).

3) Alan V. Oppenheim, Ronald W. Schafer, "Discrete-Time Signal Processing", Prentice-Hall, 3rd edition (2009).