SEL4223 Digital Signal Processing

Phase System

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Introduction

- In time-domain, phase in a signal can be seen as a time delay from the signal $\cos(\omega n)$ where

$$\cos(\omega n + \phi) = \cos\left(\omega\left(n + \frac{\phi}{\omega}\right)\right)$$

$$\sin(\omega n + \phi) = \cos(\omega n - \pi/2 + \phi) = \cos\left(\omega\left(n + \frac{\phi - \pi/2}{\omega}\right)\right)$$

- Thus, in LTI system, phase on the impulse response $(h[n])$ will delay the input signal $(x[n])$ at the output $(y[n])$

Introduction

- Prove:
- $Y(\omega) = X(\omega).H(\omega)$

$$= (X_R(\omega) + X_I(\omega))(H_R(\omega) + H_I(\omega))$$

$$= |X(\omega)|e^{j\angle X(\omega)}.|H(\omega)|e^{j\angle H(\omega)}$$

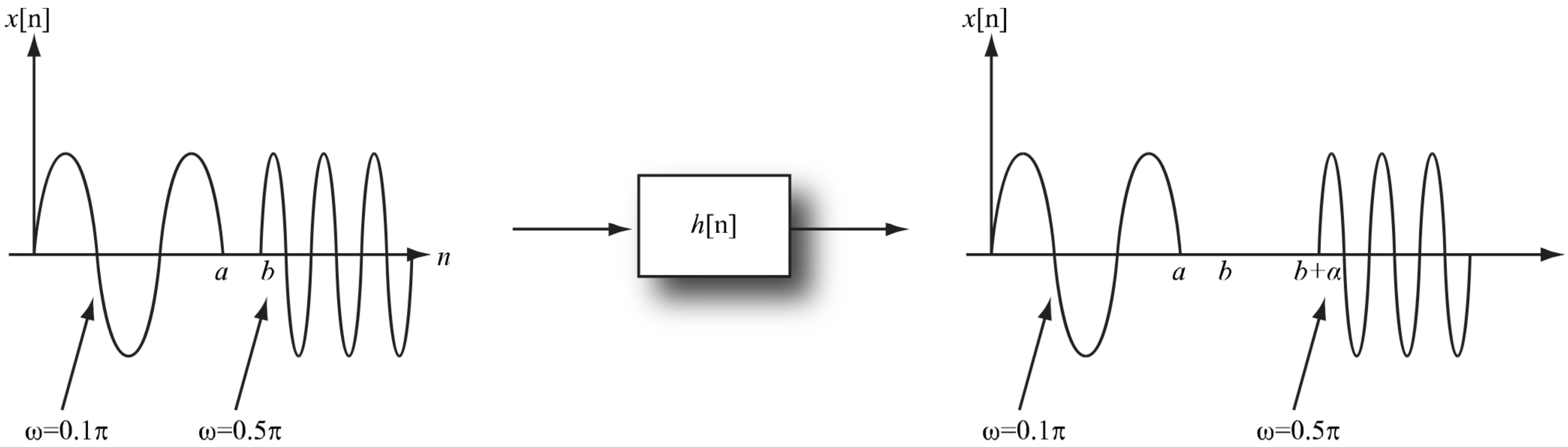
$$= e^{j\left(\angle X(\omega)+\angle H(\omega)\right)}|X(\omega)|.|H(\omega)|$$

Introduction

- Equations above shows that:
  - frequency of the output signal is obtain by multiplying magnitude response of input signal and the impulse response
  - phase of the output signal is the sum of the input signal phase and the impulse response phase.
- Thus, if the impulse response has a phase, the input signal will be delayed at the output.

Phase Distortion

- Signal can be understood as the combination of sinusoids of different frequencies. If the time delay of these frequencies is not consistent, then it can be said that the output signal has a phase distortion.



Phase Quantification

- Because phase is closely related to time delay, two measures are used to evaluate the time delay as follows:

$$Phase\ delay \quad \Longrightarrow \quad t_p = \frac{\angle H(\omega)}{\omega}$$

$$Group\ delay \quad \Longrightarrow \quad t_g = -\frac{d}{d\omega}\big(\angle H(\omega)\big)$$

- Both of the Phase delay and the Group delay are referring to amount of delay in time-domain. The difference between the two is, Phase delay is referring to delay compare to $\cos(\omega n)$ while Group delay is referring to both $\cos(\omega n)$ and $-\cos(\omega n)$. Group delay is also identified as the slope of the phase response $\angle H(\omega)$

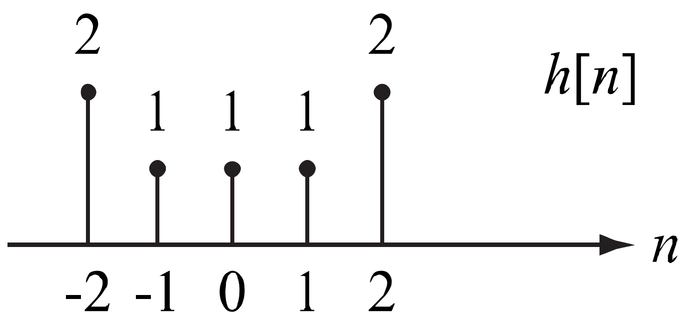
Phase System

3 types of phase response
o Zero phase system
o Linear phase system
o Non-linear phase system

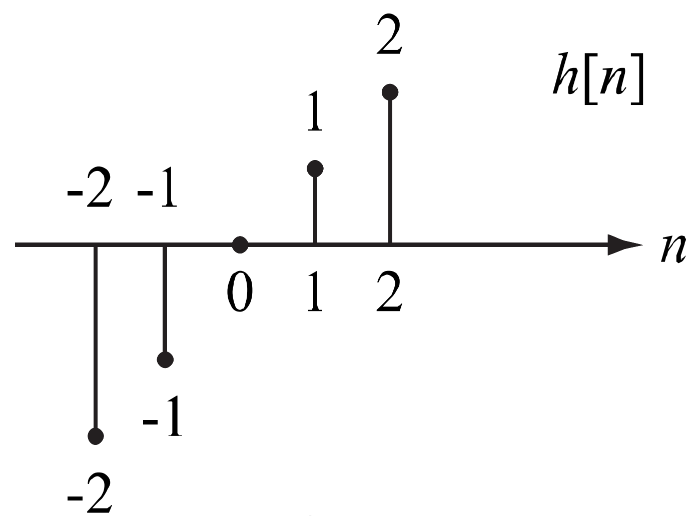
From the 3 phase response types, zero phase and linear phase do not have phase distortion while the non-linear phase has the phase distortion.

Zero Phase System $\left(t_g = 0\right)$

- $h[n]$ must be symmetry (even signal) or anti-symmetry (odd signal)
  - Symmetry (even): $h[n] = h[-n]$
  - Anti-symmetry (odd): $h[n] = -h[-n]$



Symmetry



Anti-Symmetry

Zero Phase System (cont.)

- $\angle H(\omega)$ for even and odd signal are constant values as follow:

$$\angle H(\omega)_{even} = \begin{cases} 0\ if\ H(\omega) & \geq 0 \\ \pi\ if\ H(\omega) & < 0 \\ -\pi\ if\ H(-\omega) & < 0 \end{cases}$$

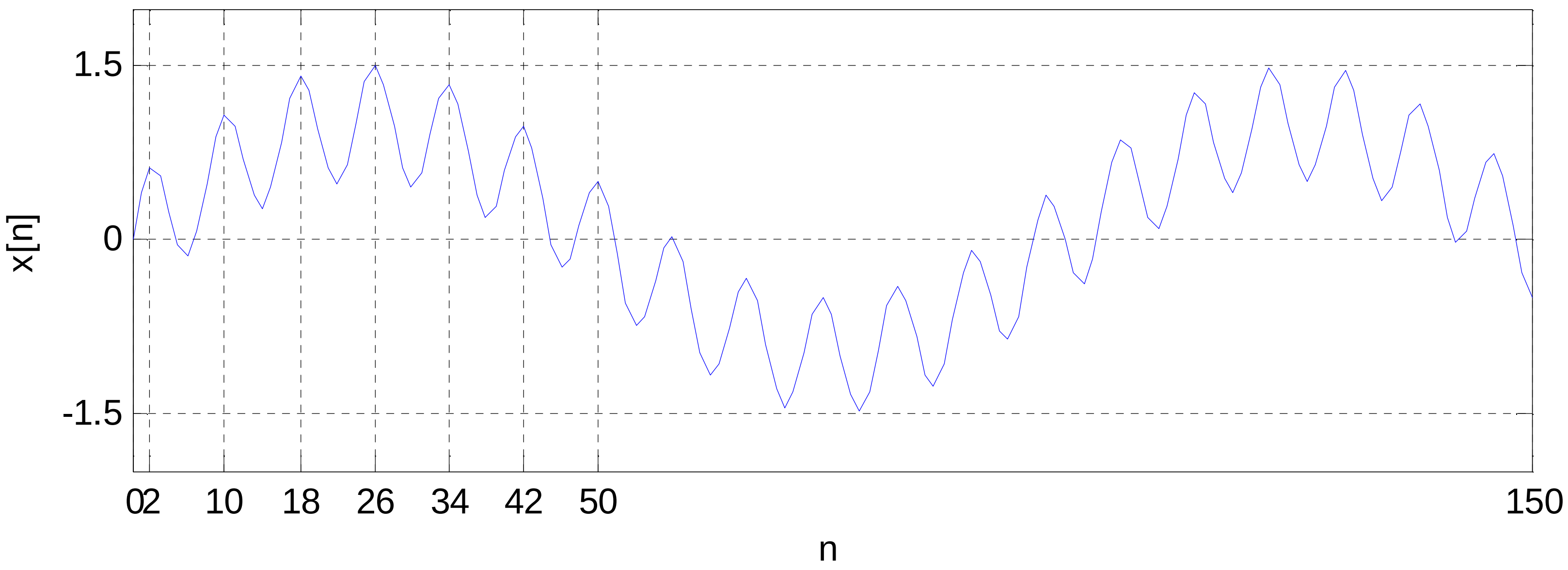
$$\angle H(\omega)_{odd} = \begin{cases} -\pi/2 & if\ \omega > 0 \\ \pi/2 & if\ \omega < 0 \end{cases}$$

Zero Phase System (cont.)

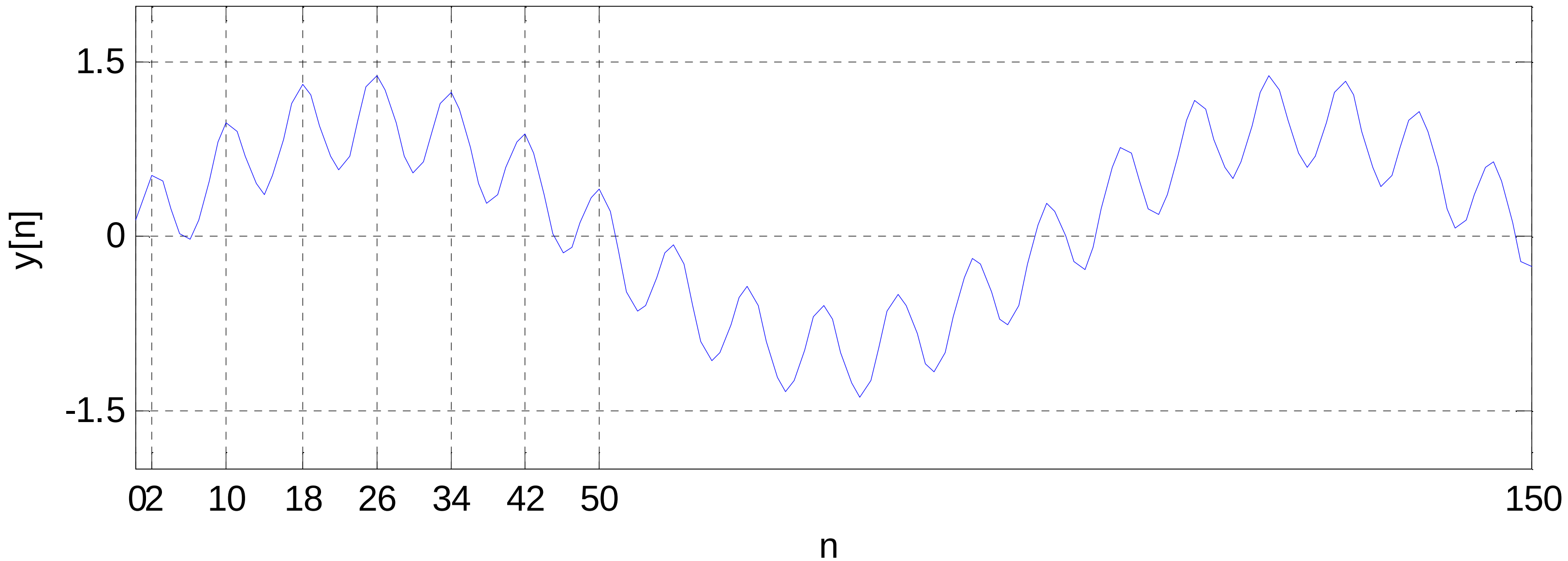
- As the $\angle H(\omega)$ is a constant value, differentiation to it results a zero value, means that all frequencies convolve with the zero phase system will have no time delay at the output.
- In other words, no phase distortion.
- The problem with zero phase system is that it is not causal where $h[n] \neq 0$ for $n < 0$. Thus, the system is not practical.

Example 1

- Find and plot $y[n] = x[n] * h[n]$ where $h[n] = \frac{1}{3}(\delta[n+1] + \delta[n] + \delta[n-1])$ and $x[n] = (0.5\sin(0.25\pi n) + \sin(0.02\pi n))(u[n] - u[n-150])$



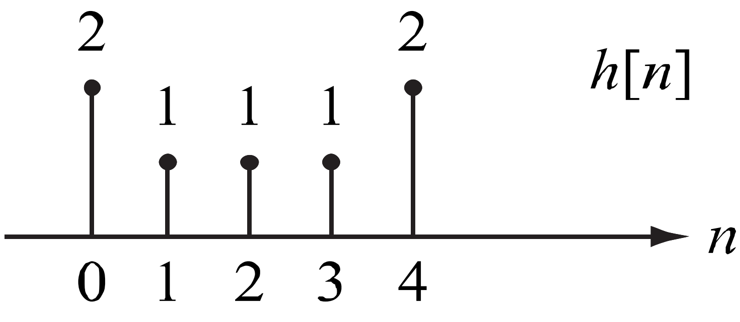
Example 1 (cont.)



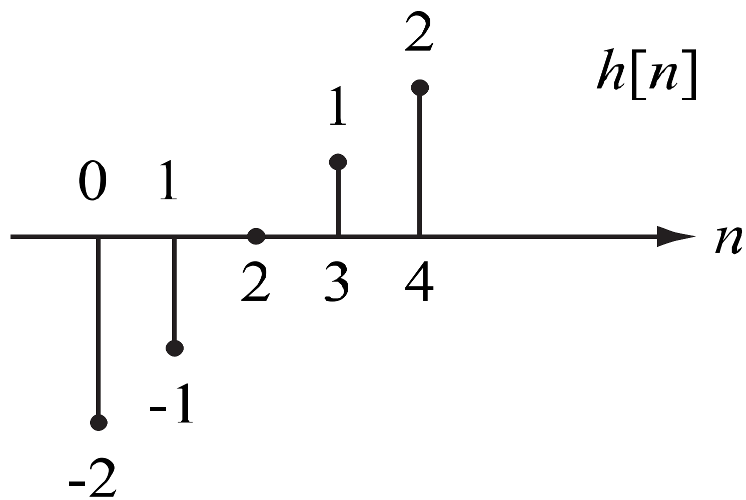
- It can be seen from Figure above that no delay occurs in the output signal.

Linear Phase System $\left(t_g = n_d\right)$

- $h[n]$ is a delayed symmetry or anti-symmetry signal



Delayed Symmetry Signal



Delayed Anti-Symmetry Signal

Linear Phase System (cont.)

- To make sure that $h[n]$ is causal, the signal is delayed so that it will have nonzero values starts at $n = 0$. Thus, the causality problem in zero phase system is solved.
- For this system,

$$\angle H(\omega) = \omega n_d \qquad \text{and} \qquad t_p = t_g = n_d$$

- Because n_d is a constant, there will be no phase distortion.
- This system is only possible for FIR system. It is not possible to have a delayed symmetry or anti-symmetry signal for IIR system.

Linear phase system variation

Type	Signal	Length $(M+1)$
I	$h[n] = h[M-n]$	Odd
II	$h[n] = h[M-n]$	even
III	$h[n] = -h[M-n]$	odd
IV	$h[n] = -h[M-n]$	even

Example 2

- Find $\angle H(\omega)$ for $h[n] = \delta[n] + \delta[n-4]$.

Solution:

- $h[n]$ can be seen as a delayed symmetry signal of $h_o[n] = \delta[n+2] + \delta[n-2]$ where $h[n] = h_o[n-2]$.
- $H_o(\omega) = h_o[0] + 2\sum_{n=1}^{1} h[n]\cos(\omega n) = 2\cos(\omega n)$.
- From the time shifting property,
- $H(\omega) = e^{-j2\omega} H_o(\omega) = 2e^{-j2\omega}\cos(\omega n)$

$$= 2\cos(\omega n)\cos(2\omega) - j2\cos(\omega n)\sin(2\omega)$$

Example 2 (cont.)

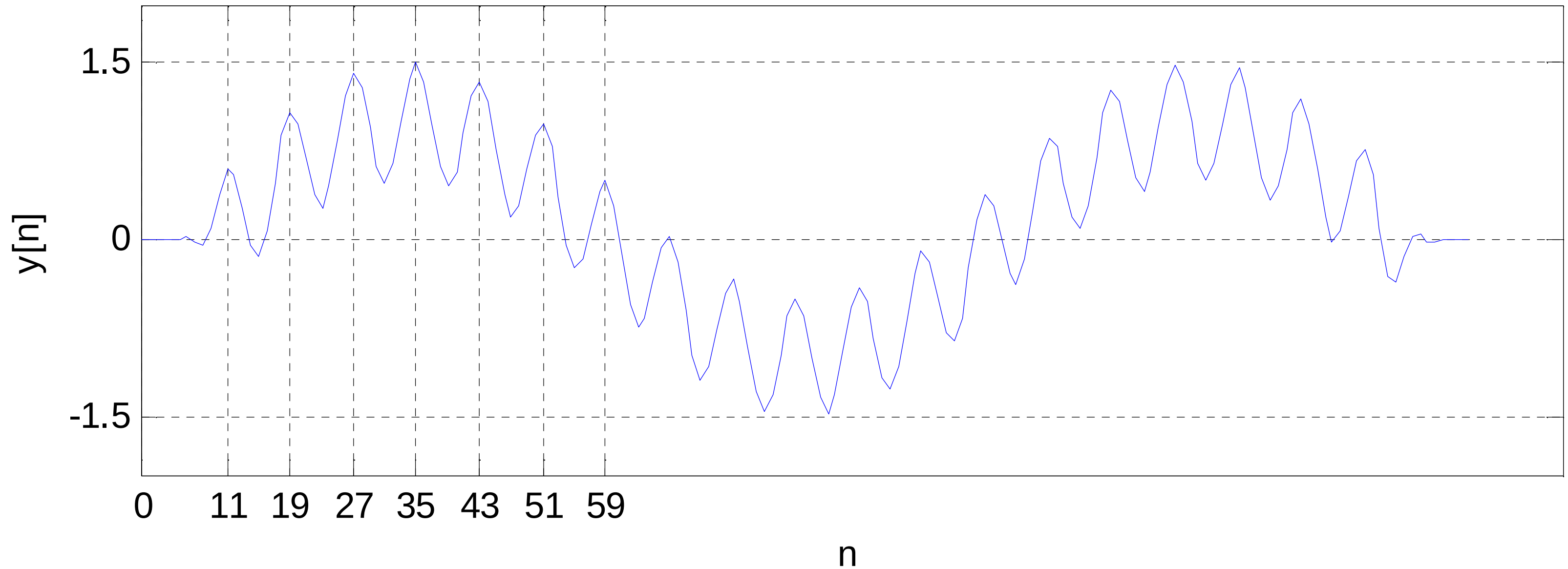
- Thus,
- $\angle H(\omega) = \tan^{-1} - \frac{2\sin(2\omega)}{2\cos(2\omega)} = -2\omega$
- This shows that the system is linear phase system with $\angle H(\omega) = \omega n_d$

  Where $n_d = -2$

Example 3

- Repeat Example 1 but with impulse response below:
- $h[n] = [0.0036,\ 0,\ -0.0123,\ 0,\ 0.0344,\ 0,\ -0.0860,\ 0,\ 0.3111,\ 0.5,\ 0.3111,\ 0,\ -0.0860,\ 0,\ 0.0344,\ 0,\ -0.0123,\ 0,\ 0.0036]$
- In this example, $h[n]$ is a delayed symmetry signal with 9 samples delay. Basically $h[n]$ will remove all frequencies greater than 0.5π. Thus, both frequencies in $x[n]$ will be preserved at the output.
- Because $h[n]$ is delayed 9 samples from its symmetry version, $\angle H(\omega) = -9\omega$

Example 3 (cont.)



- By looking at peaks of the output signal, it can be seen that the two frequencies, $\omega = 0.25\pi$ and $\omega = 0.02\pi$ were both delayed by 9 samples.

Non-Linear Phase System

- $h[n]$ can be both FIR and IIR
- $h[n]$ can also be both stable and causal
- Phase delay and Group delay vary throughout the entire frequency band.
- Other than zero-phase and linear phase systems, the system is considered as non-linear phase system.
- The problem is, it suffers phase distortion.

Example 4

- Repeat Example 3 but with 4th order IIR Butterworth filter as below:
- $H(\omega) =$
$$\frac{\left(1+e^{-j\omega}\right)^4}{\left(9.5578-11.1851e^{-j\omega}+5.6273e^{-j2\omega}\right)\left(12.337-11.1851e^{-j\omega}+2.8481e^{-j2\omega}\right)}$$
- From the $H(\omega)$, difference equation of the system is
- $$\begin{aligned} y[n] = 0.0085(x[n] + 4x[n-1] + 6x[n-2] + 4x[n-3] \\ +x[n-4] + 244.89y[n-1] - 221.75y[n-2] \\ +94.8y[n-3] - 16.03y[n-4]) \end{aligned}$$
- Similar to filter in Example 3, this filter will also pass through both frequencies at $\omega = 0.02\pi$ and $\omega = 0.25\pi$.

Example 4 (cont.)

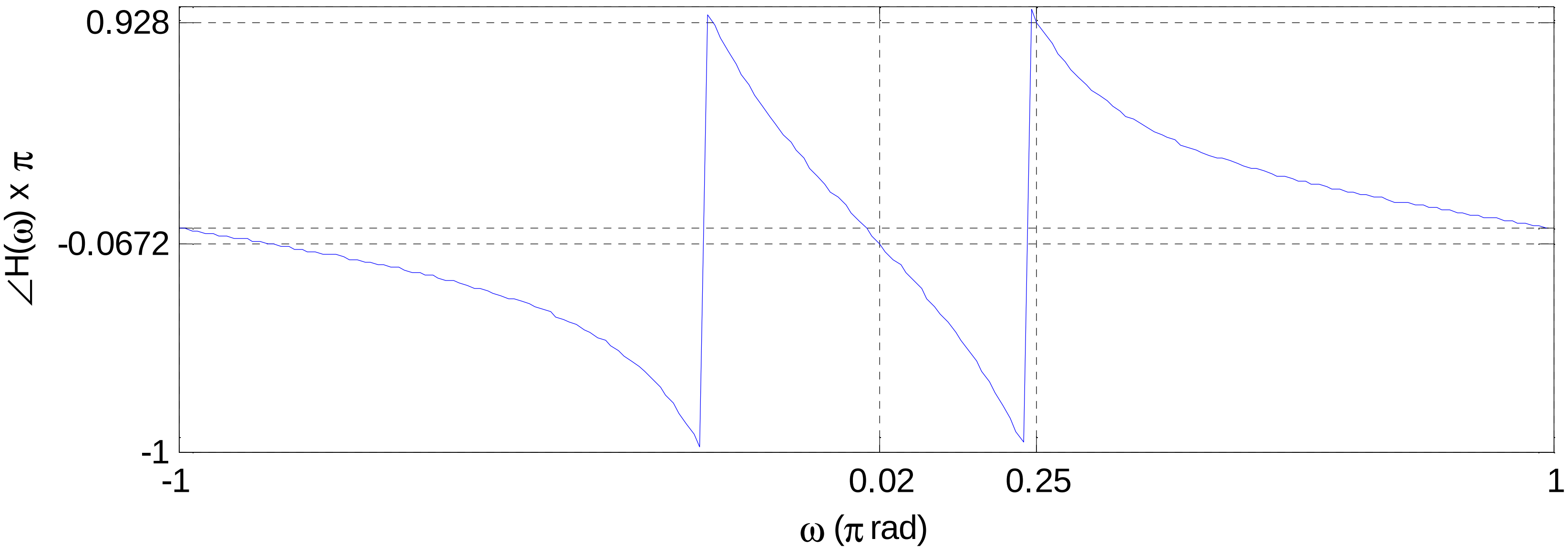
- Phase response $\angle H(\omega)$ of the filter is shown in next figure where $\angle H(0.02\pi) = -0.0672$ and $\angle H(0.25\pi) = 0.928$. Other than that, the figure obviously shows a non-linear plot as the slope of the plot varies.

- Phase delay at the two input frequencies can be computed as:

- $t_p(0.02\pi) = -\frac{0.0672\pi}{0.02\pi} = -3.36,$ $t_p(0.25\pi) = \frac{0.928\pi}{0.25\pi} = 3.712$

- This means that signal with frequency 0.02π is shift 3.36 samples to the right while signal with frequency 0.25π is shift 3.712 samples to the left. Because the delays are different between the two frequencies, thus phase distortion occurs.

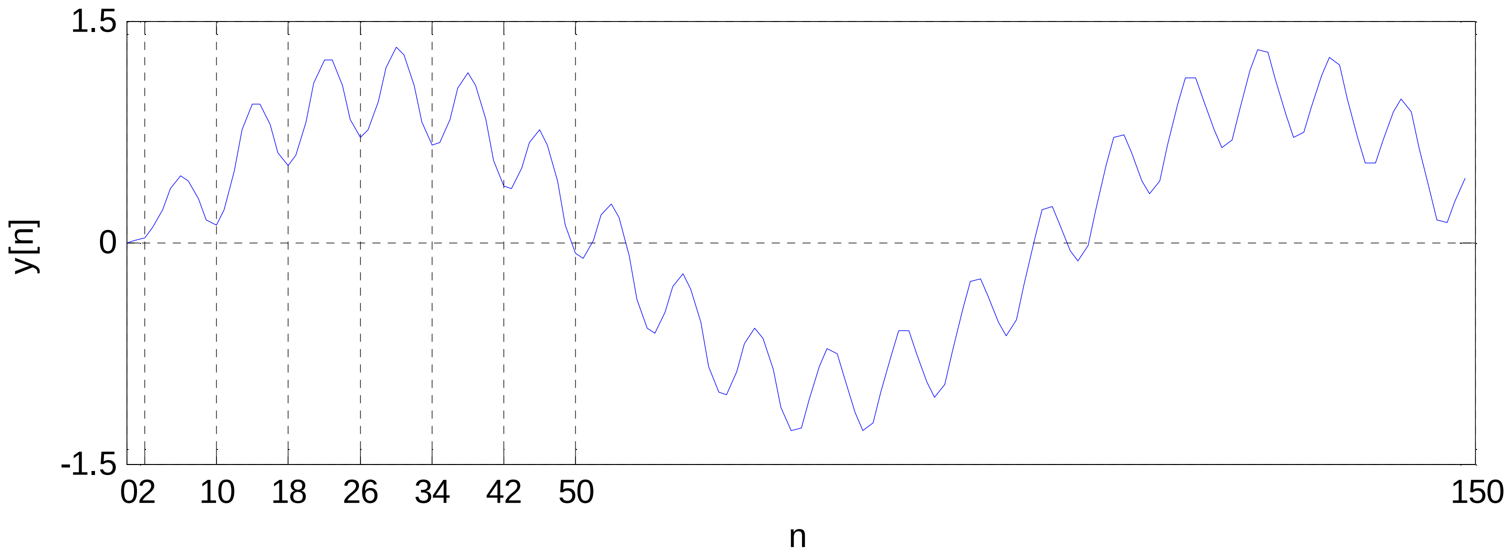
Example 4 (cont.)



- Figure above shows the results where peaks for frequency 0.25π positions in the input signal have moved significantly while delay for frequency 0.02π is unnoticeable.

Example 4 (cont.)

- Delay of the frequency 0.25π is obvious because it only need 8 samples to complete one period while the delay for the frequency is almost half of it which is 3.712 samples.
- For frequency 0.02π, although the delay is almost the same with delay of the frequency 0.25π which is 3.36 samples, this is still too small compare to the 100 samples needed to complete its one period.



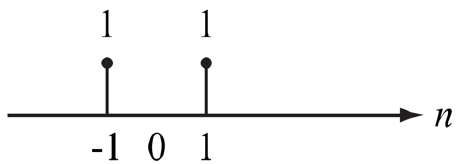
Quiz

Determine whether the signal has zero phase, linear phase or non-linear phase.

1. $h[n] = u[n]$
2. $h[n] = u[n] + u[-n]$
3. $h[n] = u[n] + u[-n-1]$
4. $h[n] = \delta[n] + 2\delta[n]$
5. $h[n] = \delta[n] - 2\delta[n-1] + 2\delta[n+1]$
6. $h[n] = a^{|n|}, a < 1$
7. $h[n] = a^{|n|}u[n], a < 1$
8. $y[n] = y[n-1] + x[n]$
9. $h[n] = u[n] - u[-n]$
10. $y[n] = 2y[n-1] + x[n] - 2x[n-1]$

Plot the pole-zero plot, magnitude response and frequency response of these signals

1)



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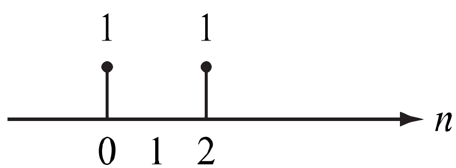

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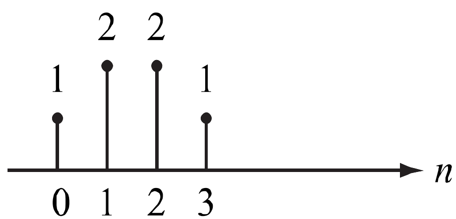

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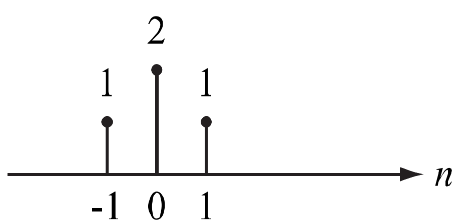
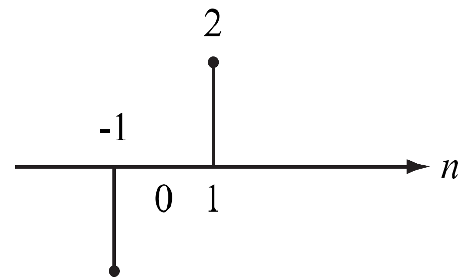
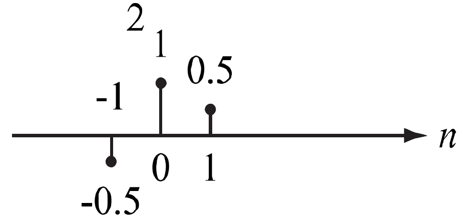


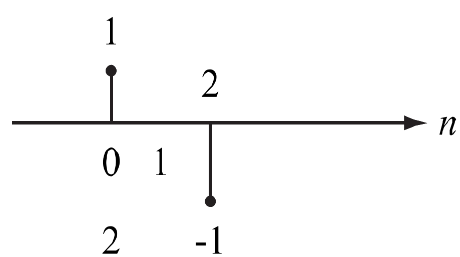
5)



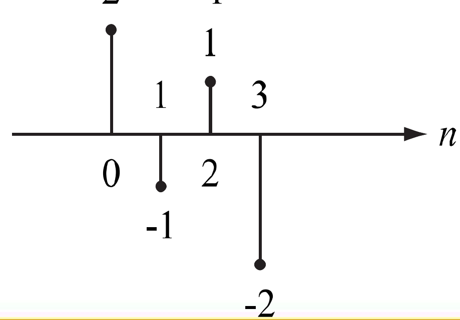
6)



7)



8)

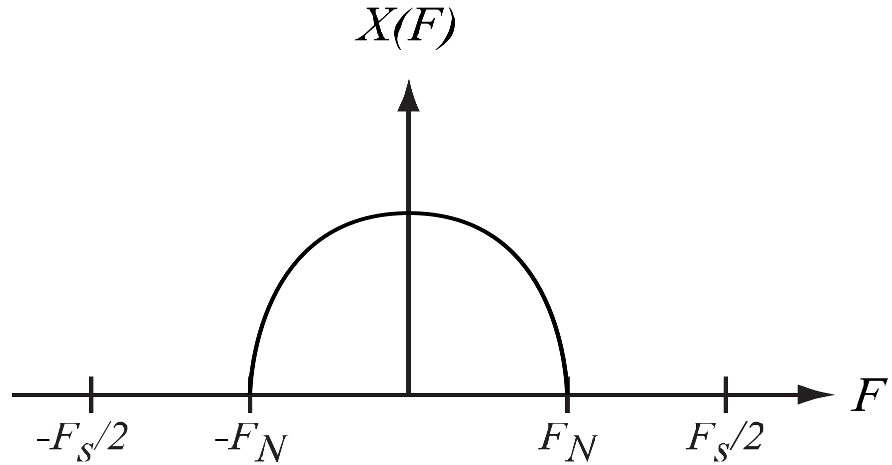


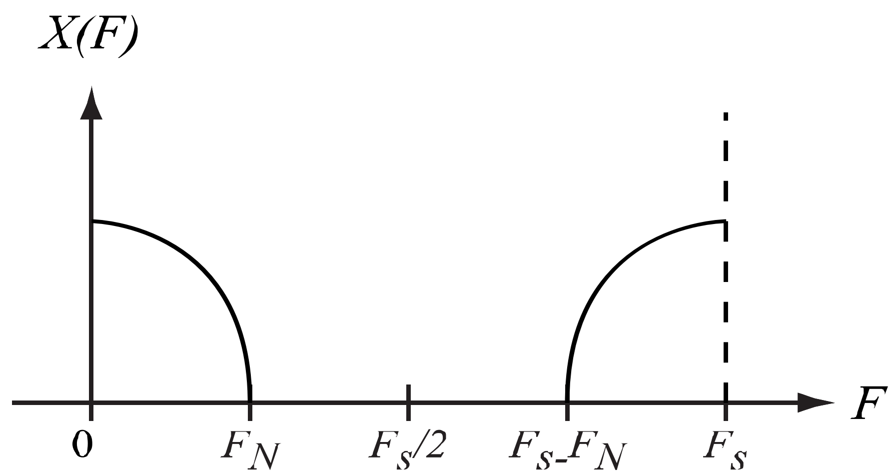
Sampling Revisited

- A process of converting analog signal to discrete signal
- $x(t) = x_c(nT_s), t = nT_s$
- In frequency domain, $\omega = 2\pi f = \frac{2\pi F}{F_s}$.
- At $\omega = \pi$, where it is the end frequency for discrete signal, $F = {}^{F_s}/_{2}$. Thus,

Frequency components preserved after the sampling are the frequencies less than ${}^{F_s}/_{2}$

Sampling Revisited (cont.)



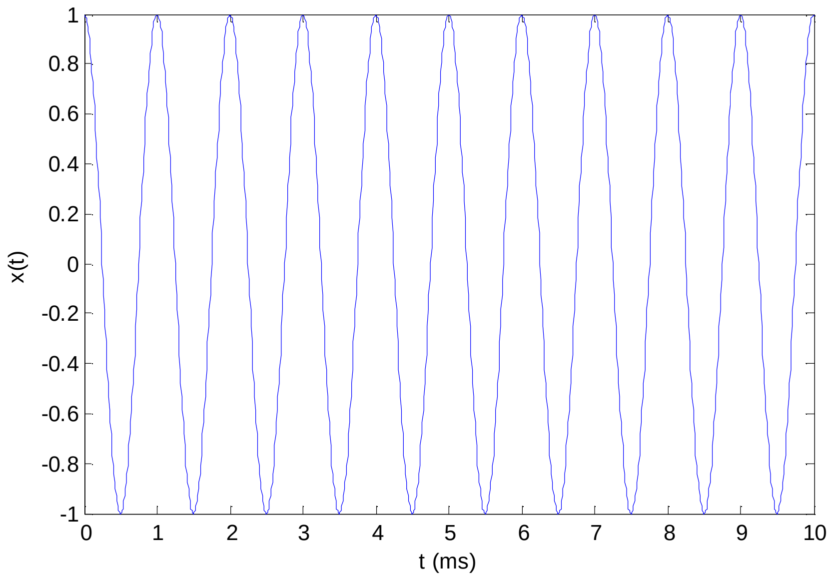


Example 5

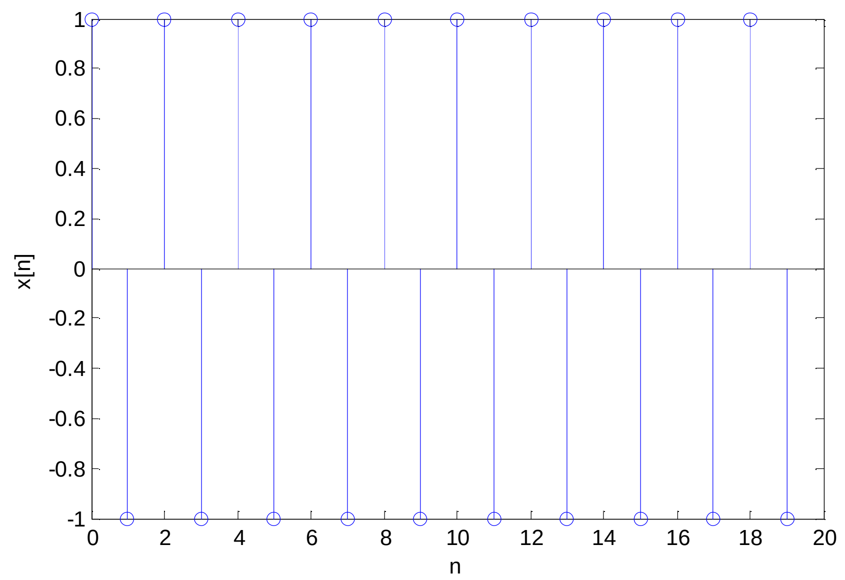
- $x(t) = \cos(2\pi 1000t)$ for $0 \leq t < 10ms$
- If $F_s = 2000\text{Hz}$
- $x[n] = \cos\left(\frac{2\pi 1000n}{2000}\right) \qquad for \;\; 0 \leq n < \frac{10ms}{0.5ms}$

  $= \cos(\pi n) = \cos(\omega_c n) \qquad for \;\; 0 \leq n < 20$
- $\omega_c = \pi$
- Thus, when $F_s = 2000\text{Hz}$, the signal frequency of $\Omega = 2\pi 1000$ is mapped at $\omega = \pi$

Example 5 (cont.)

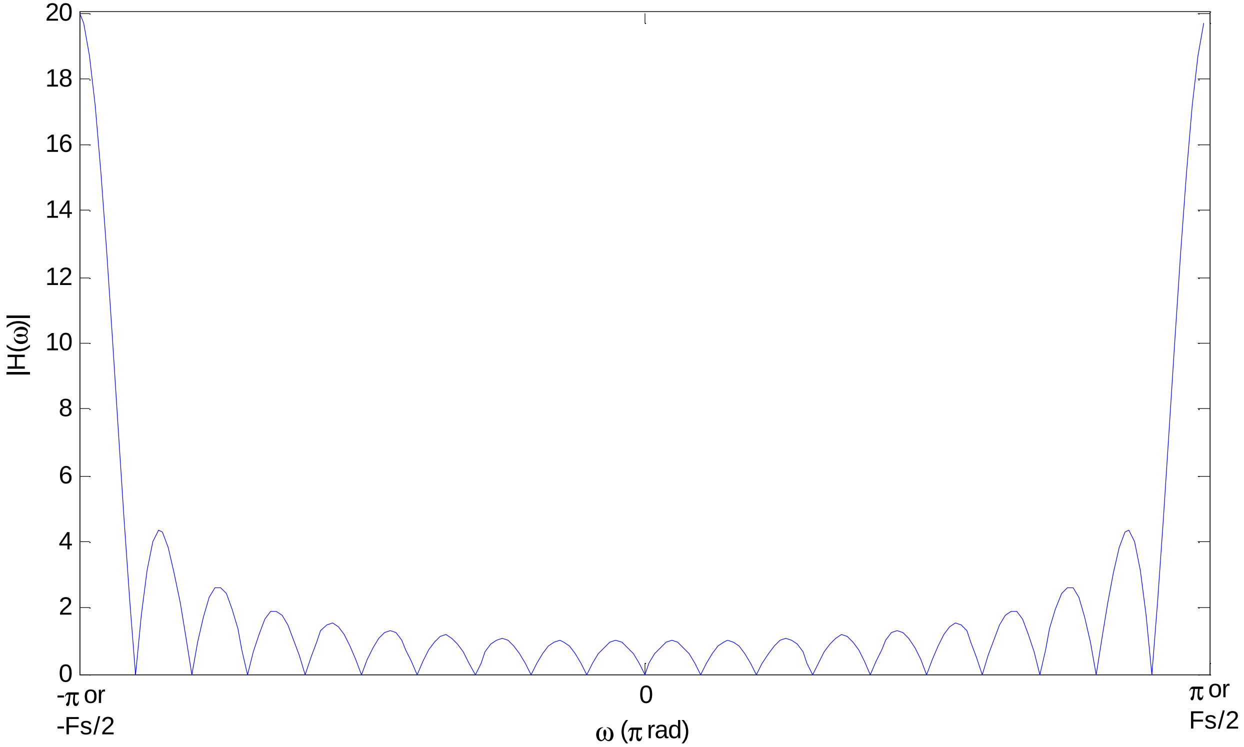


Continuous signal



Discrete signal sampled at $F_s = 1000Hz$

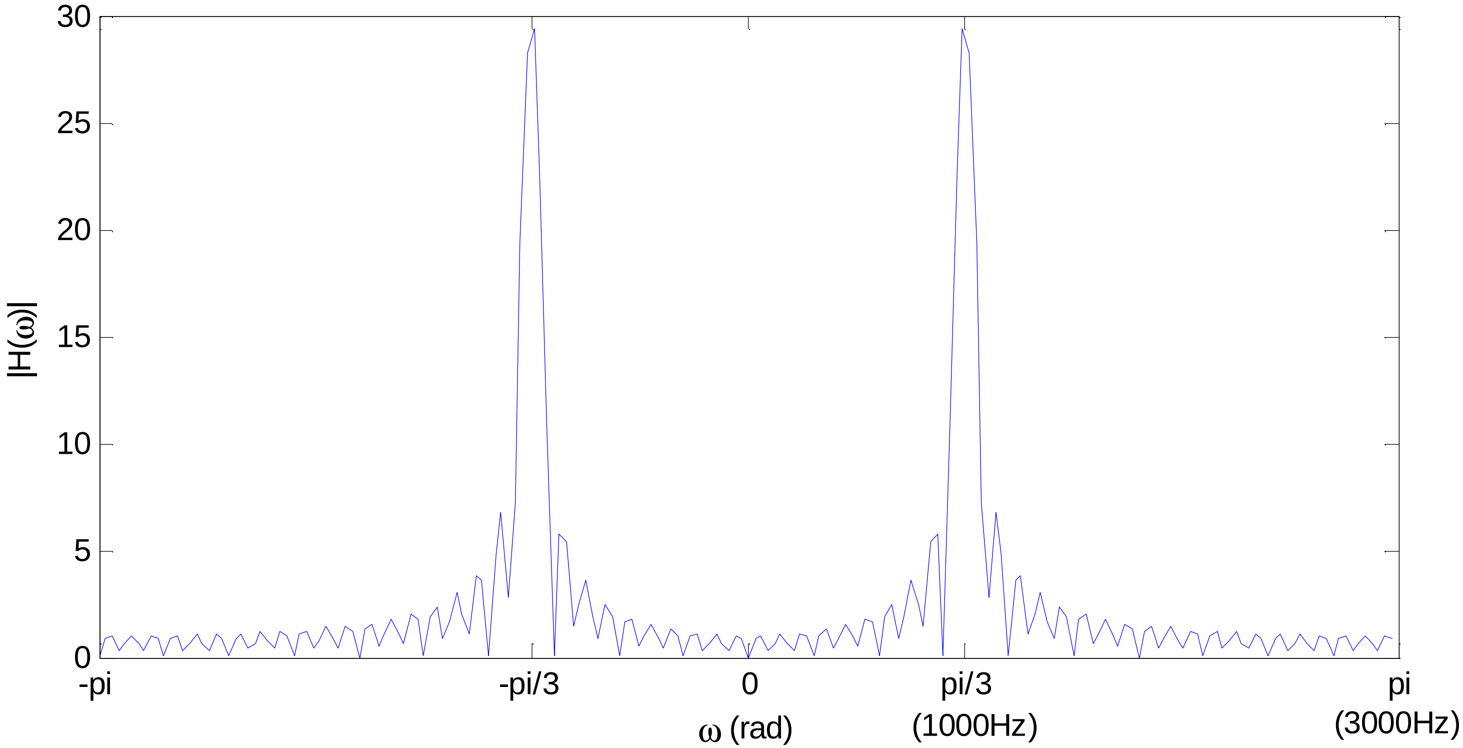
Example 5 (cont.)



Magnitude response - frequency $\Omega = 2\pi 1000\; rads^{-1}$ is mapped at $\omega = \pi\; rad$

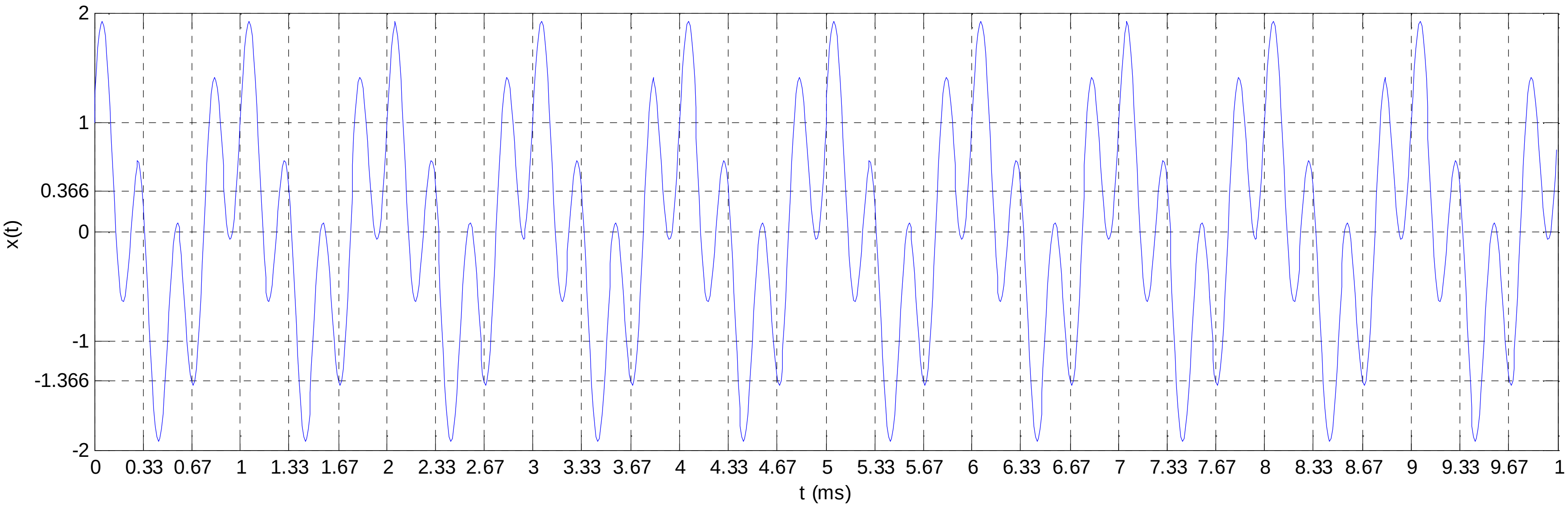
Example 5 (cont.)

- When $F_s = 6000\text{Hz}$, the signal frequency $\Omega = 2\pi 1000$ is mapped at $\omega = \frac{1}{3}\pi$



Example 6

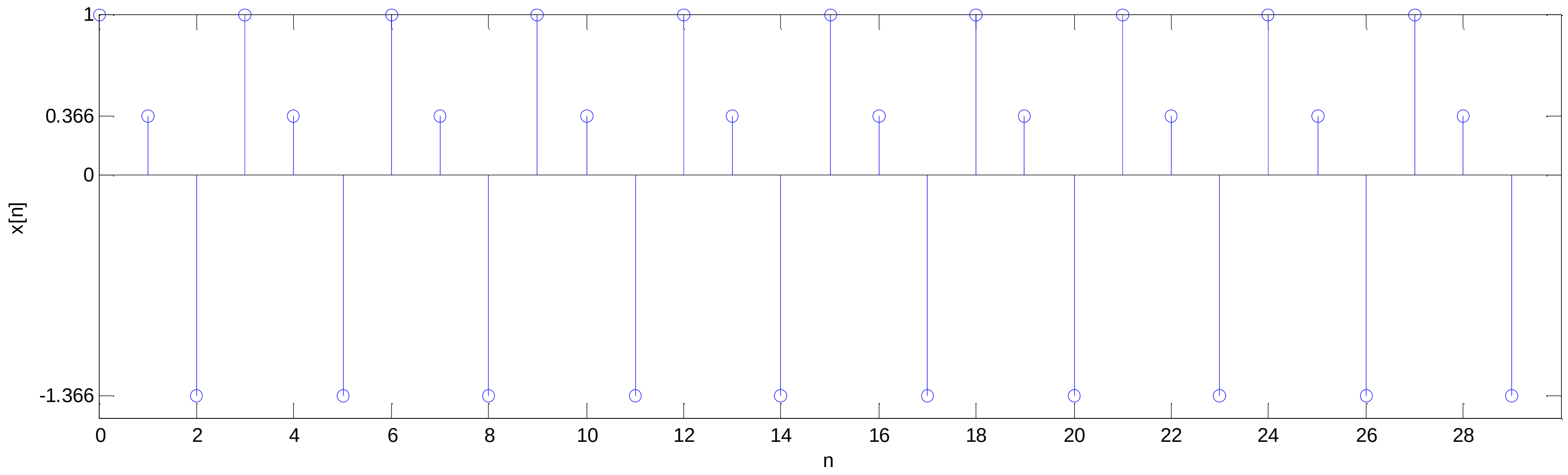
- $x(t) = \cos(2\pi 1000t) + \sin(2\pi 4000t)$ for $0 \leq t < 10ms$



Example 6 (cont.)

For $F_s = 3000$Hz,

Number of sample in $x[n]$ is $N = \frac{10ms}{T_s} = \frac{10ms}{0.3333ms} = 30$



Example 6 (cont.)

- In equation, the discrete signal is written as:
- $x[n] = \cos\left(\frac{2\pi 1000n}{3000}\right) + \sin\left(\frac{2\pi 4000n}{3000}\right) \quad for \;\; 0 \leq n < 10ms \times 3000Hz$
- $x[n] = \cos\left(\frac{2\pi n}{3}\right) + \sin\left(\frac{8\pi n}{3}\right)$

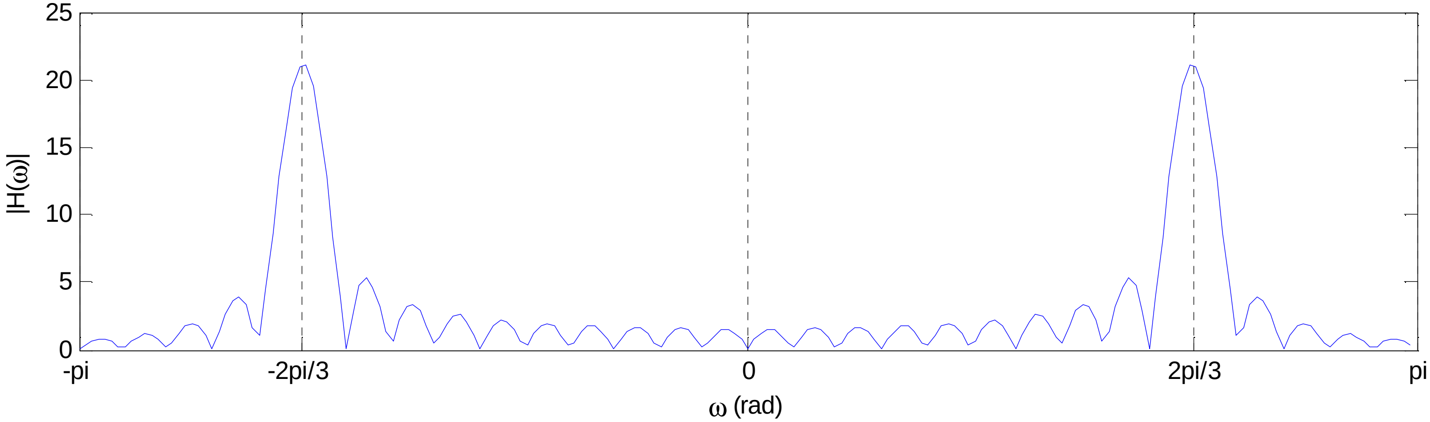
$$= \cos\left(\frac{2\pi n}{3}\right) + \sin\left(2\frac{2}{3}\pi n\right)$$

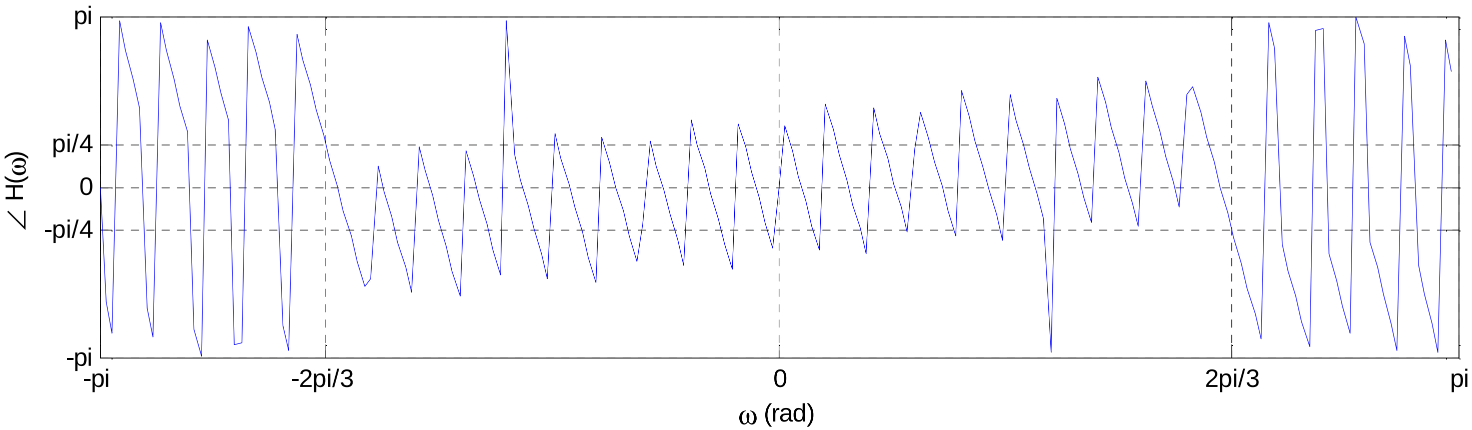
$$= \cos\left(\frac{2\pi n}{3}\right) + \sin\left(\frac{2}{3}\pi n\right) \qquad for \quad 0 \leq n < 30$$

Example 6 (cont.)

- It is shown that when $F_s = 3000$Hz, both frequency of the 1000Hz and the 4000Hz of the continuous signal are mapped to $\omega = \frac{2}{3}\pi$.
- In other words, although there are two frequencies exist in the continuous signal, only one frequency appears in its discrete form.
- The following figures show the frequency response of the discrete signal.

Example 6 (cont.)

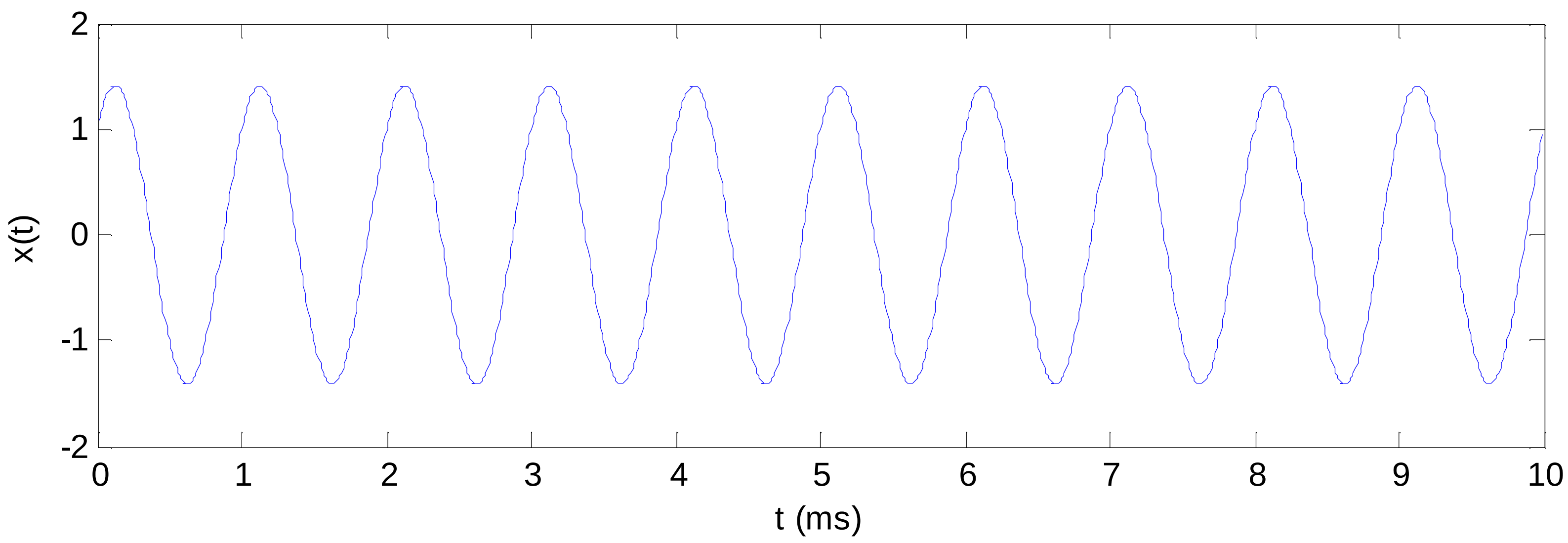




Example 6 (cont.)

- Thus, when $x[n]$ is converted back to its continuous signal, the frequency component of 4000Hz is missing as shown in equation and figure below. This phenomenon is called ‘**aliasing**’.

$$x(t) = \cos(2\pi 1000t) + \sin(2\pi 1000t)$$



Example 6 (cont.)

- To obtain a good sampling output, $F_s \geq 2F_N$, where F_N is the maximum frequency in the original continuous signal. This is called the **Nyquist Theorem**

- If we now choose $F_s = 10000\text{Hz} \geq 2F_N \geq 8000\text{Hz}$,

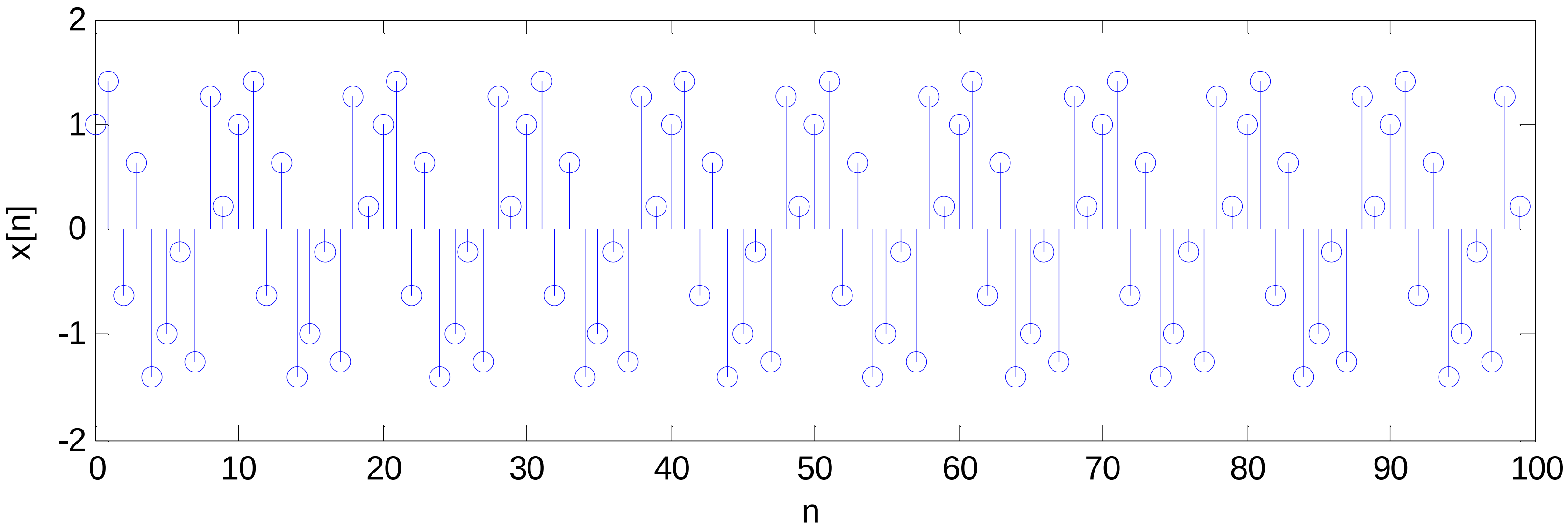
- $x[n] = \cos\left(\frac{2\pi 1000n}{10000}\right) + \sin\left(\frac{2\pi 4000n}{10000}\right) \quad for \quad 0 \leq n < 10ms \times 10kHz$

$$= \cos\left(\frac{\pi}{5}n\right) + \sin\left(\frac{4}{5}\pi n\right) \qquad for \quad 0 \leq n < 100$$

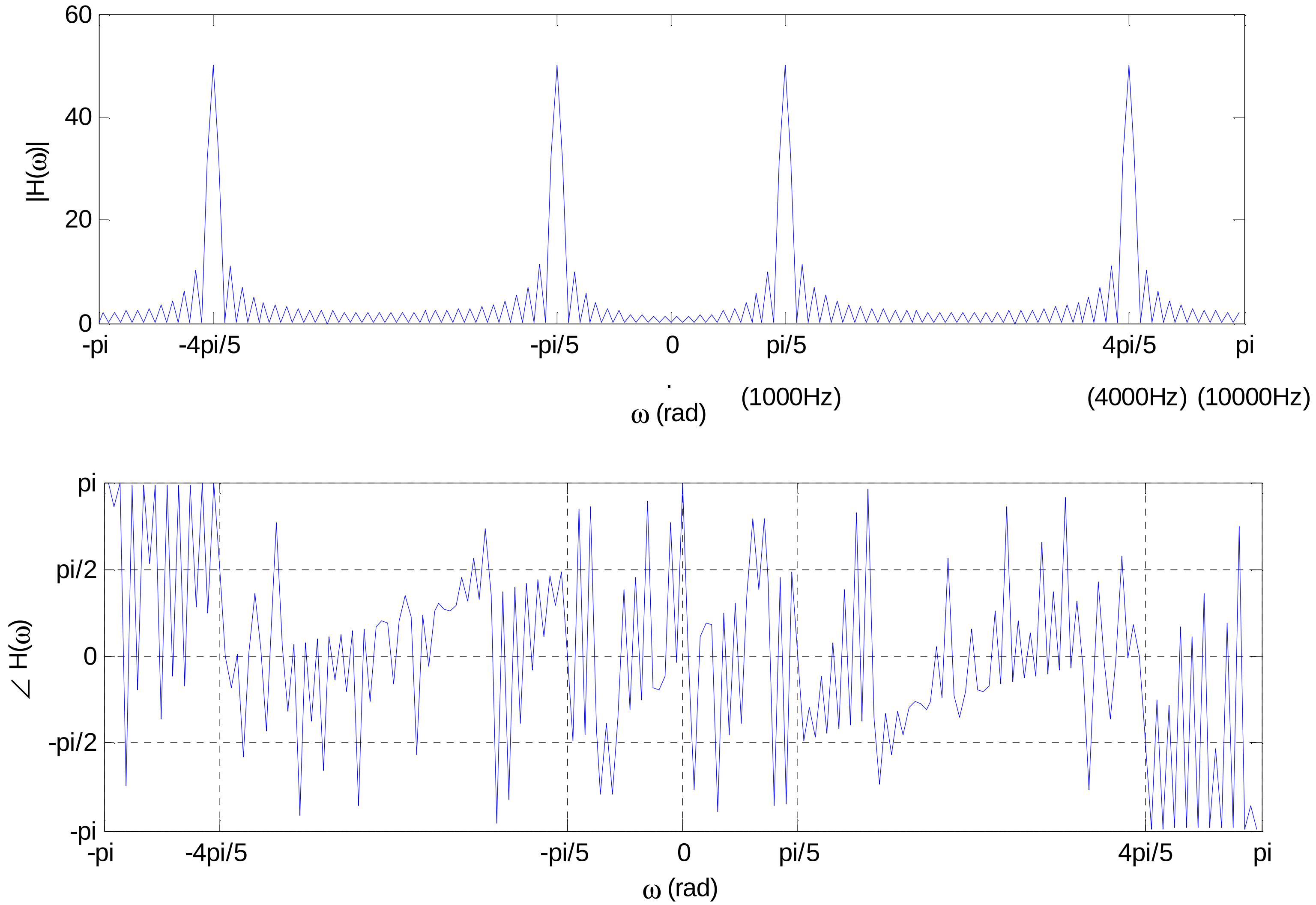
- Now, frequency of 1kHz is mapped to $\omega = \frac{\pi}{5}$ and frequency of 4kHz is mapped to $\omega = \frac{4\pi}{5}$. This shows that both of the frequencies in the continuous-time signal are preserved where no aliasing occur.

Example 6 (cont.)

- The following figures show the new discrete time domain signal and its frequency spectrum respectively.



Example 6 (cont.)



References

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2) Sanjit K. Mitra, "Digital Signal Processing-A Computer Based Approach", McGraw-Hill Companies, 3rd edition (2005).

3) Alan V. Oppenheim, Ronald W. Schafer, "Discrete-Time Signal Processing", Prentice-Hall, 3rd edition (2009).