

CHAPTER 4

Response and Stability Analysis in Time Domain

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Chapter Outline

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- System category, Poles, zeros, order and type
- System response

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- Response Specifications

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- Second Order Response
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- Steady state error for unity feedback

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4.1 Introduction

This chapter focuses on analysis the response and the stability of a control system in time domain for first order and second order systems. The performance and steady state error of the control system under specified conditions will be analyzed. Furthermore, for higher order system, the stability and system performance will be identified using Routh Hurwitz(R-H) method.

In general, unity feedback control system can be described as Figure 4.1.

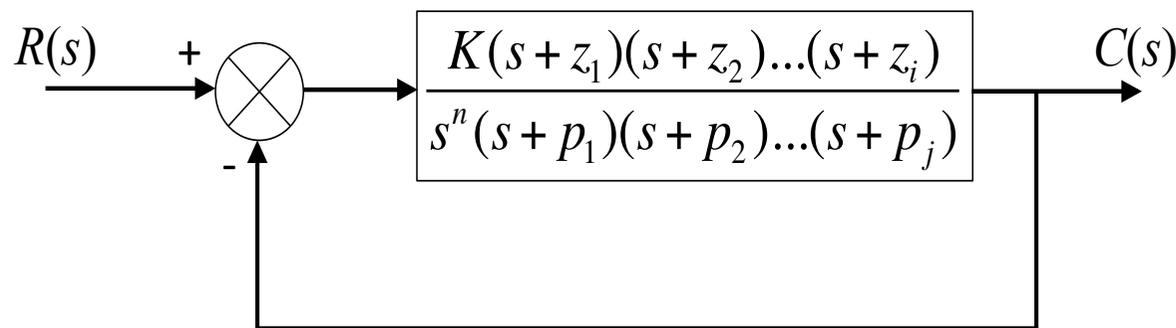


Figure 4.1 Unity feedback control system

- K : Gain of the system
- z_1, z_2, \dots, z_i : Zeros of the system
- p_1, p_2, \dots, p_j : Poles of the system
- n : $n = 1, 2, 3 \dots$ Represents type of the system

- **System Order** is the order of the denominator of the transfer function after cancellation of the common factors in the numerator.
- **System Type** is the value of n in the denominator or the number of pure integration in the forward path,
 - $n=0$: type 0
 - $n=1$: type 1 and so on

Poles (x): the value of the Laplace transform variable, s , that cause the transfer function to **become infinite**.

Zeros (o): the value of the Laplace transform variable, s , that cause the transfer function to **become zero**.

System Response :

- System response is the output response, which consists of the sum of *forced response* and *natural response*.
- *Forced response* is contributed by the input signal and the natural response is from the nature of the system.
- *Natural response* determines how good the system is.

- The *transient response* as shown in Figure 4.2, is the change in output response from the beginning of the response to the final state of the response and the steady state response is the output response as time is approaching infinity (or no more changes at the output).

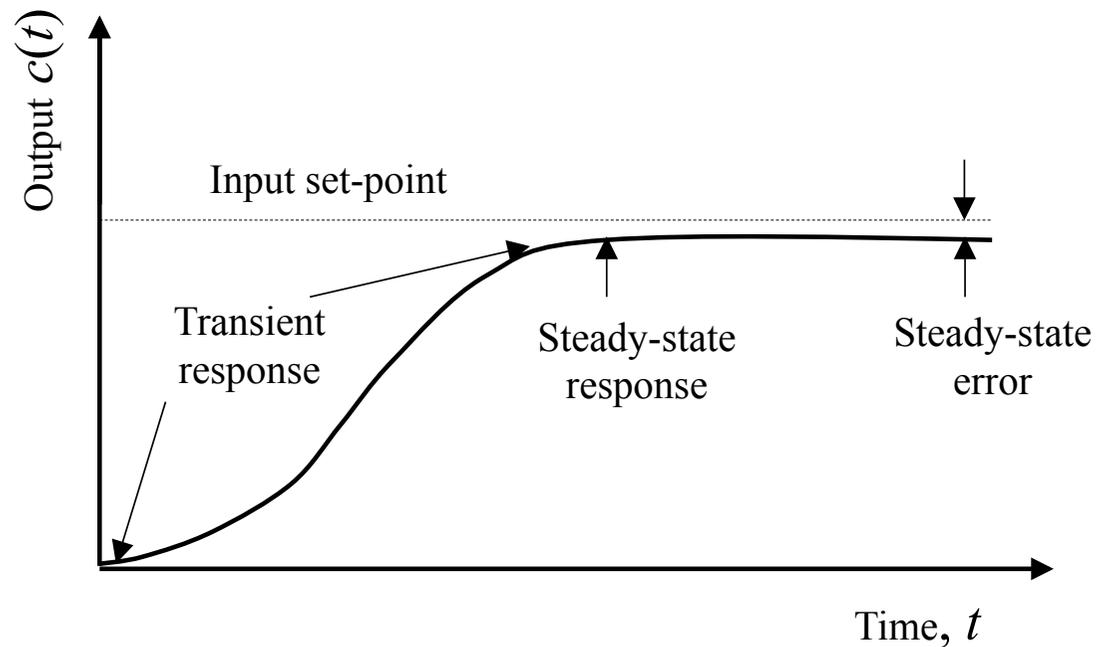


Figure 4.2 System response

Example 1: Plot poles and zeros of the following system. The system's transfer function:

$$\frac{C(s)}{R(s)} = \frac{s + 3}{s + 7}$$

Then find the $c(t)$

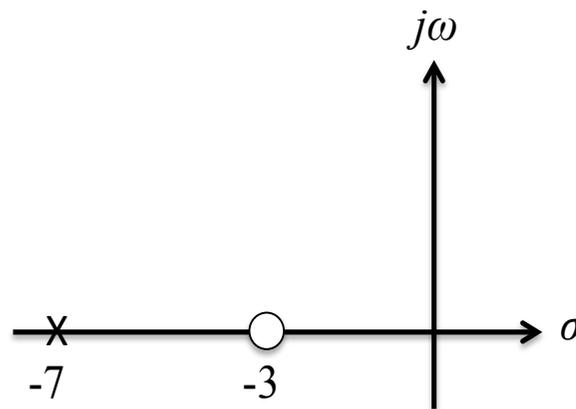
Solution:

We can plot the pole and zero of the system on s-plane as follows:

Pole: $s = -7$

Zero: $s = -3$

Figure 4.3 s-plane plot



When a unit step function $R(s)$ is applied to the system:

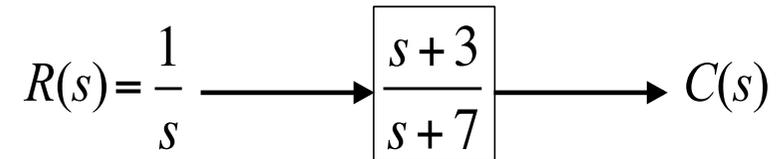


Figure 4.4 System transfer function

Substitute this input into the transfer function and applying the partial fraction

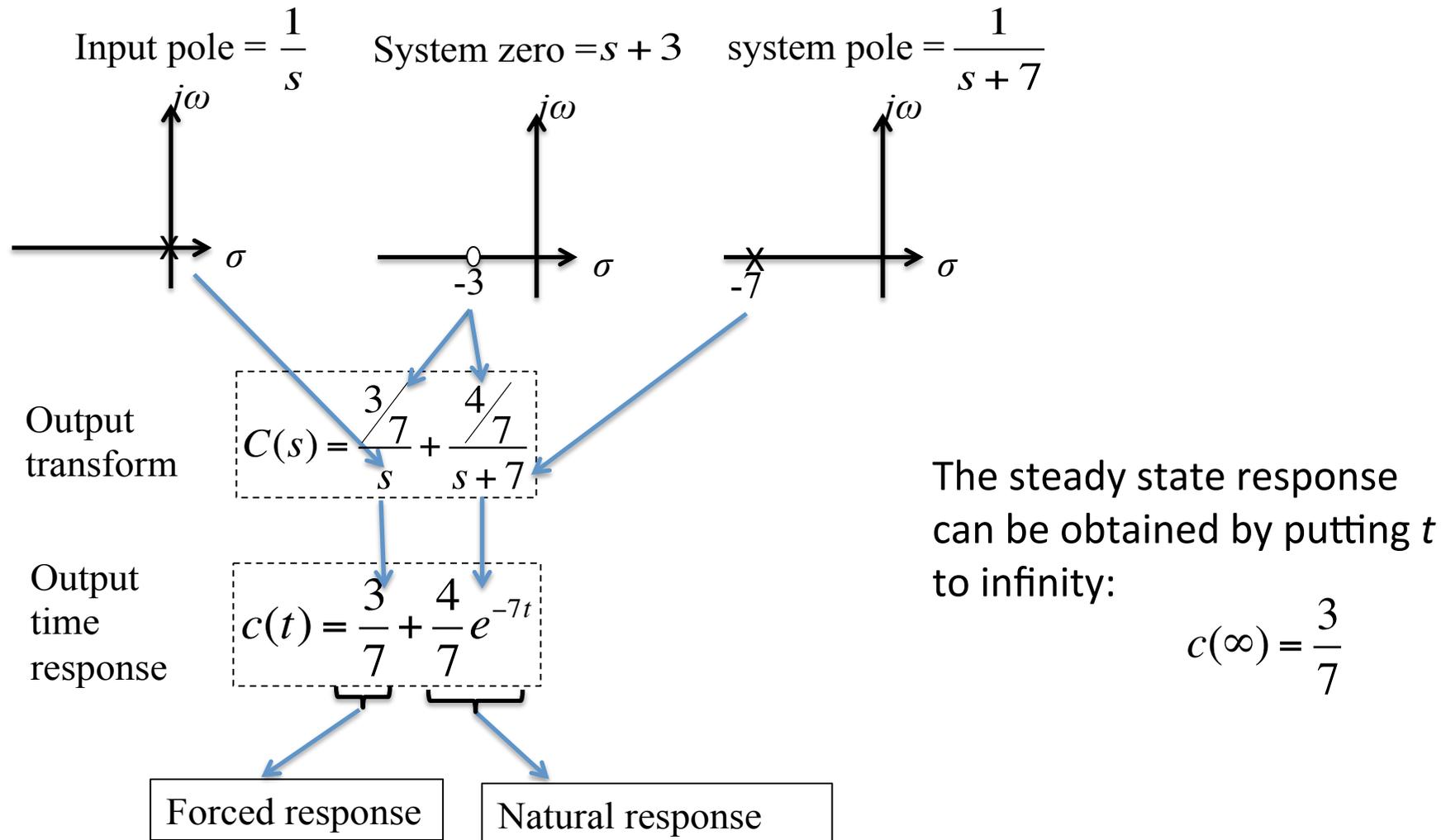
$$C(s) = \frac{s+3}{s+7} \cdot \frac{1}{s} = \frac{A}{s} + \frac{B}{s+7}, \quad \Rightarrow A = \frac{3}{7}, \quad B = \frac{4}{7}$$

$$C(s) = \frac{3}{7} \cdot \frac{1}{s} + \frac{4}{7} \cdot \frac{1}{s+7}$$

Applying the inverse Laplace transform, gives the output response

$$c(t) = \frac{3}{7} + \frac{4}{7} e^{-7t}$$

The output response shows the transient response of the system as the time is varied.



Based on the output response, in general, we can write the output response $c(t)$ of any system as follows:

- a) Any input to a system will give the forced response at the output
- b) The poles in the transfer function of a system will give the natural response at the output
- c) The poles in the transfer function of a system will give the term, which is associated with the pole location on s -plane. In this case, , where $-\alpha = -7$. The further away (to the negative side of the real axis) the pole location (α), the faster the term goes to 0.
- d) The combination of zeros and poles of a system contributes to the magnitude of the output response

Example 4.2: Given the system with unit step input as

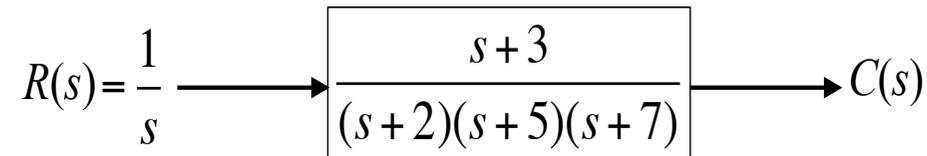


Figure 4.5 Example 4.2

Solution:

By inspection, in general the output partial fraction is:

$$C(s) = \frac{K_1}{s} + \frac{K_2}{s+2} + \frac{K_3}{s+5} + \frac{K_4}{s+7}$$

Taking the inverse Laplace transform:

$$c(t) = \underbrace{K_1}_{\substack{\text{Force} \\ \text{response}}} + \underbrace{K_2 e^{-2t} + K_3 e^{-5t} + K_4 e^{-7t}}_{\text{Natural response}}$$

4.2

First order system

4.2.1 Transient & steady state response

This section discusses first-order control system without zeros, which can be described by transfer function shown in Figure 4.7. Test input signal is unit step in order to define the performance specifications of such system

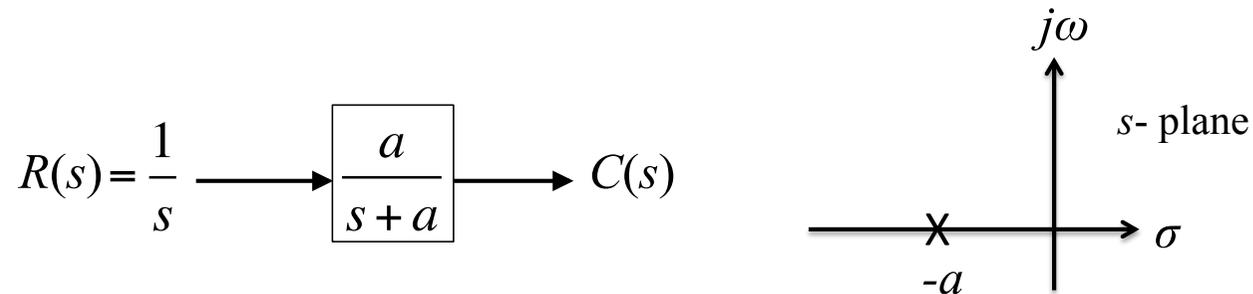


Figure 4.6 First-order system without zeros

Laplace transform of the step response is:

$$C(s) = \frac{1}{s} \cdot \frac{a}{s+a} \quad \longrightarrow \quad c(t) = c_f(t) + c_n(t) = 1 - e^{-at} \quad ; \text{ for } t \geq 0$$

$$\text{When } t = 1/a: \quad \longrightarrow \quad c(1/a) = 1 - e^{-a(1/a)} = 1 - 0.37 = 0.63$$

The significance of parameter a , to describe the transient response of this system. When $t = 1/a$

$$c(t)\Big|_{t=1/a} = 1 - e^{-a(1/a)} = 1 - 0.37 = 0.63$$

Try for $t = \frac{2}{a}; \frac{3}{a}; \frac{4}{a}; \frac{5}{a}$ and plot the response $c(t)$ versus t :

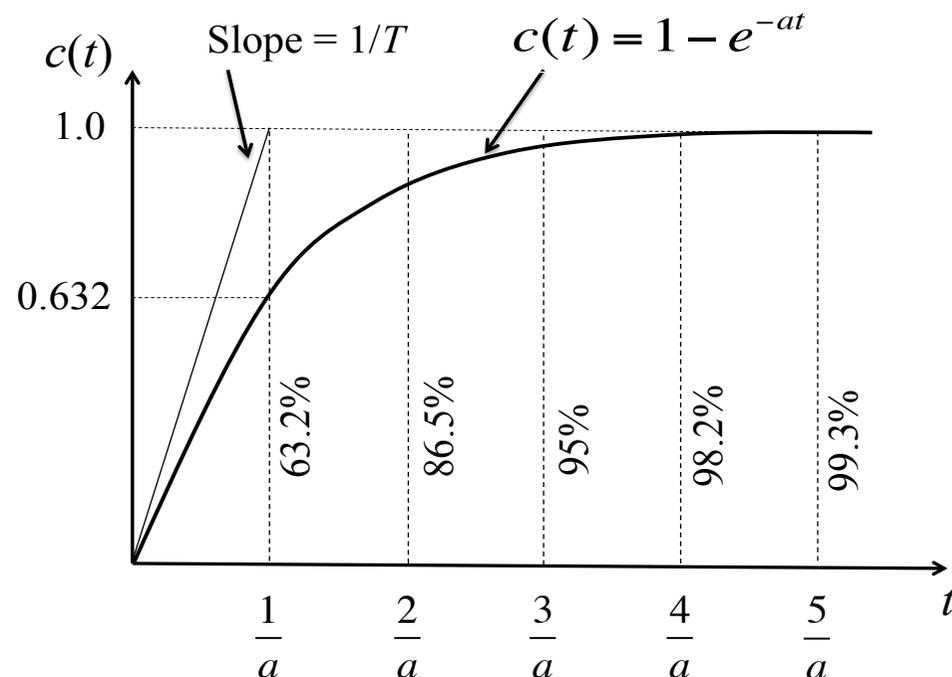


Figure 4.7 1st order system response and specifications

4.2.2 Response Specifications

Several definitions and specifications can be obtained from the output response of a first order system Figure 4.8:

- a) Time constant (τ)
- b) Rise time (T_r)
- c) Settling time (T_s)

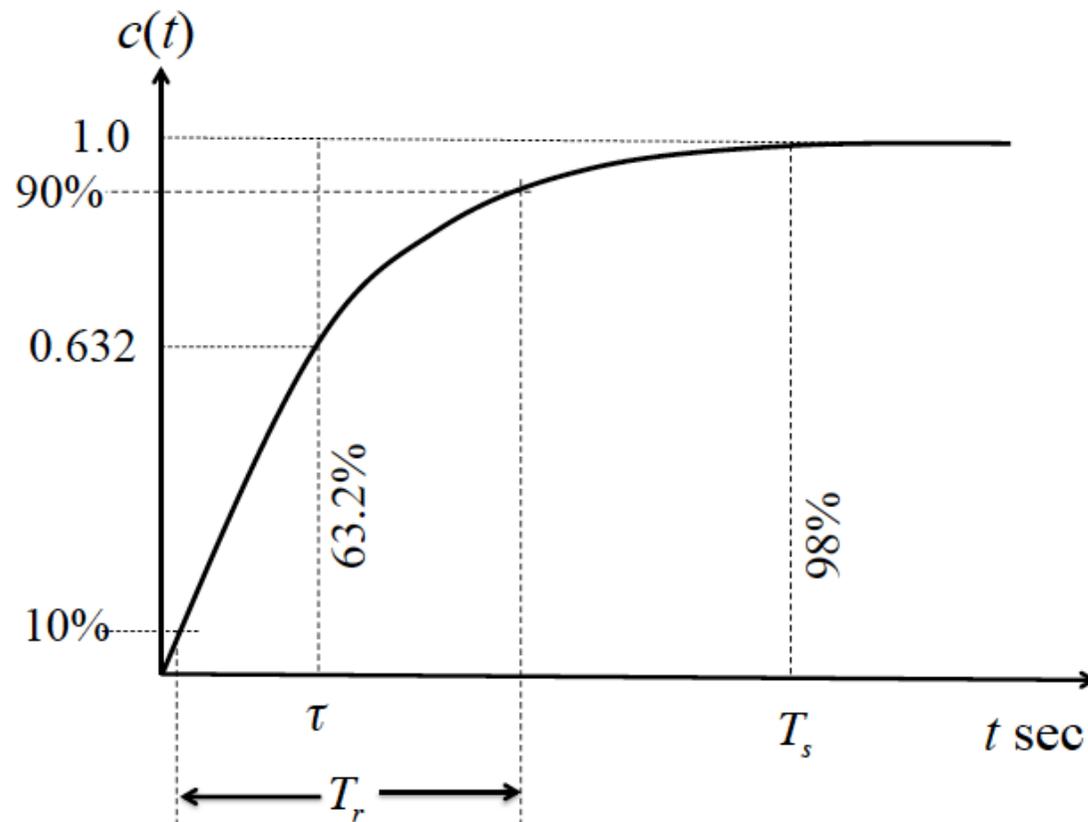


Figure 4.8

a) Time constant, τ

- Is the time for step response to rise to 63% of its final value (speed of the system responds to the step input)
- Is the time for e^{-at} to decay to 37% of initial value.
- Parameter a is exponential frequency (unit 1/sec)
- Time constant (τ) shows the speed of the system responds to the step input
- The further to the left a pole (a) is on the $-ve$ real axis, the faster the exponential transient response will decay to zero

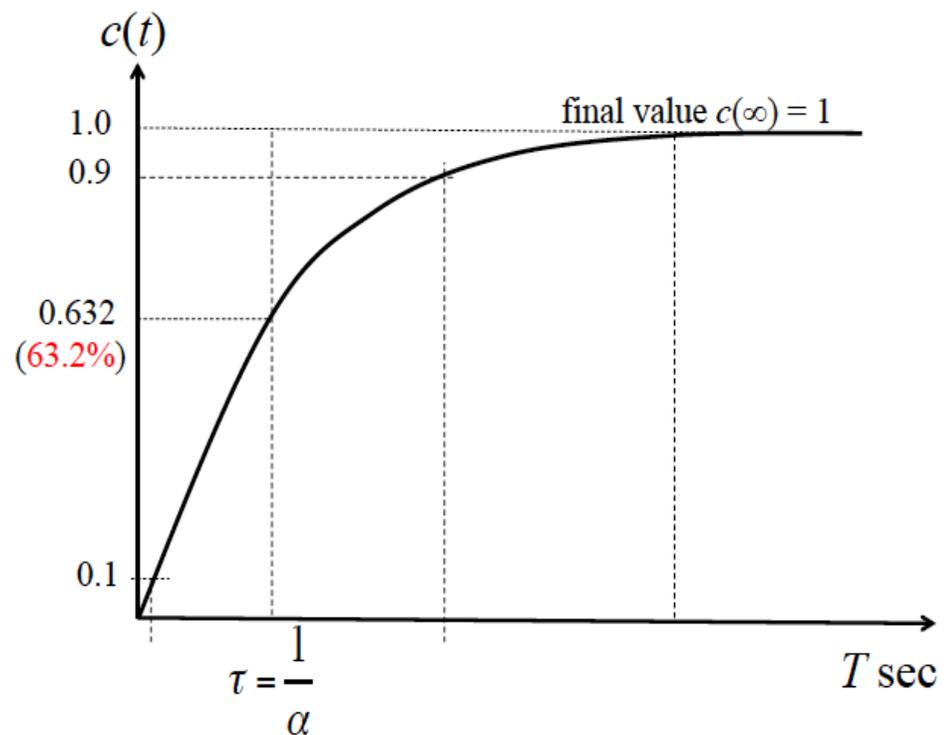


Figure 4.9

b) Rise Time, T_r

- is the time for the step input waveform to go from 10% to 90% of its final value
- $T_r = [\text{Time at } c(t) = 90\% \text{ of final value; } 0.9] - [\text{Time at } c(t) = 10\% \text{ of final value ; } 0.1]$

$$T_r = \frac{2.31}{a} - \frac{0.11}{a} = \frac{2.2}{a}$$

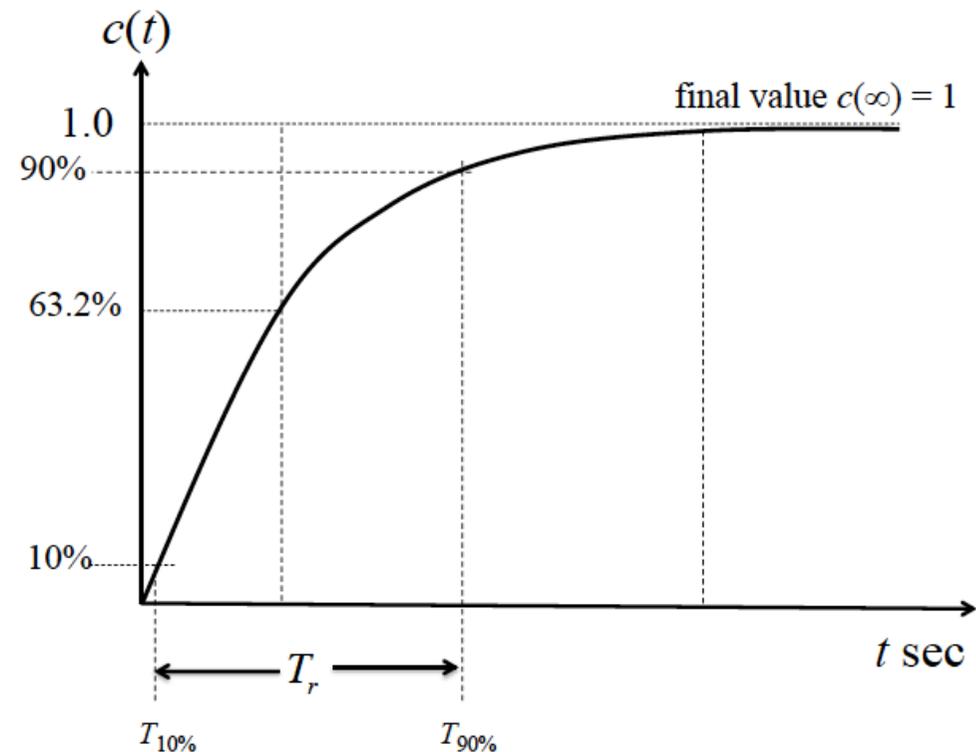


Figure 4.10

c) Settling Time, T_s

Settling time (T_s) is the time for the response to reach and stay within $\pm 2\%$ (or $\pm 5\%$) from of its final value. For the case accuracy is $\pm 2\%$ of unit step input ($1/s$), at $t = T_s$, output $c(t)$ is within values 0.98 and 1.02 .

$$c(T_s) = 1 - e^{-aT_s} = 0.98$$

$$T_s = \frac{4}{a}$$

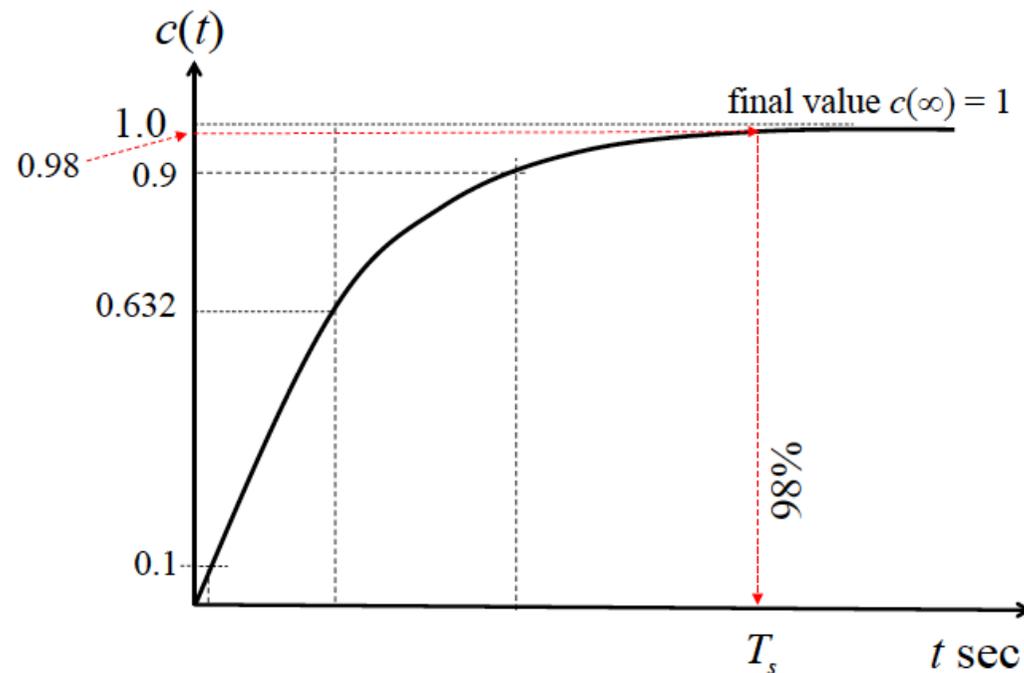


Figure 4.11

Example 4.3: First order unit step response is given below. Identify time constant (τ), settling time T_s , and rise time T_r

$$\frac{C(s)}{R(s)} = \frac{10}{s+10}$$

Solution:

Time constant (τ) =

$$\frac{1}{a} \Rightarrow \frac{C(s)}{R(s)} = \frac{a}{s+a}$$

$$\rightarrow a = 10, \tau = 0.1s$$

$$T_r = \frac{2.2}{a} = \frac{2.2}{10} = 0.22s$$

$$T_s = \frac{4}{a} = \frac{4}{10} = 0.4s$$

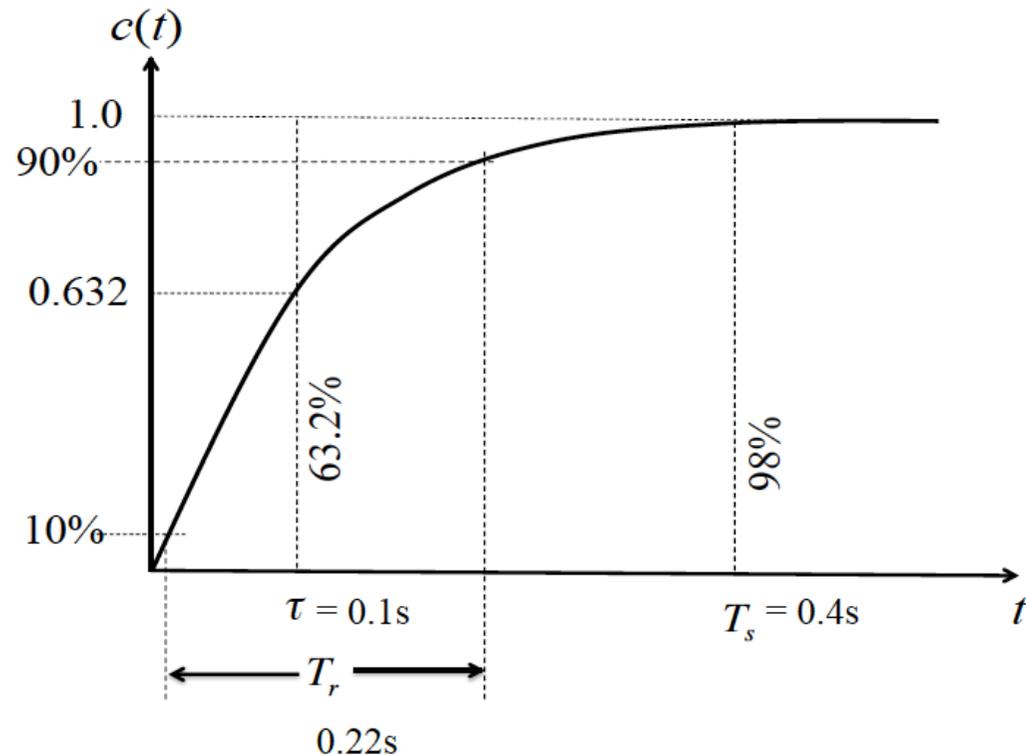


Figure 4.12

4.3

Second order system

This section discusses the response of second-order (2nd) control system to a step input function. In general second order system has two poles and its transfer function can be defined as

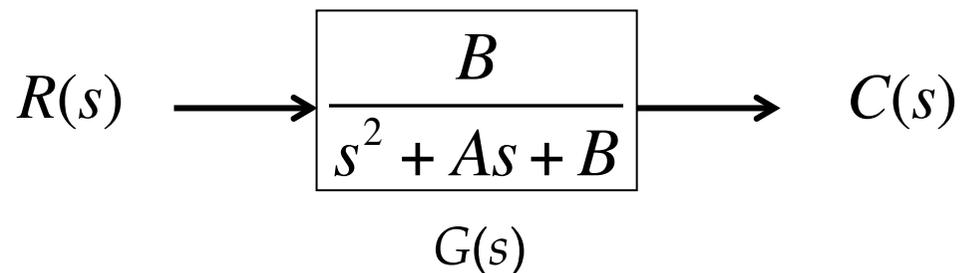


Figure 4.13

System transfer function:

$$\frac{C(s)}{R(s)} = \frac{B}{s^2 + As + B}$$

4.3.1 Poles Location Versus Step Response

System's response can be classified into four categories based on the poles location on s-plane:

- a) Overdamped response
- b) Critically damped response
- c) Under damped response
- d) Undamped response

a) Overdamped response

Transfer function:
$$\frac{C(s)}{R(s)} = \frac{16}{s^2 + 10s + 16}$$

Poles location on s-plane:

$$\frac{C(s)}{R(s)} = \frac{16}{(s+2)(s+8)} \rightarrow \text{poles are at } s=-2, s = -8$$

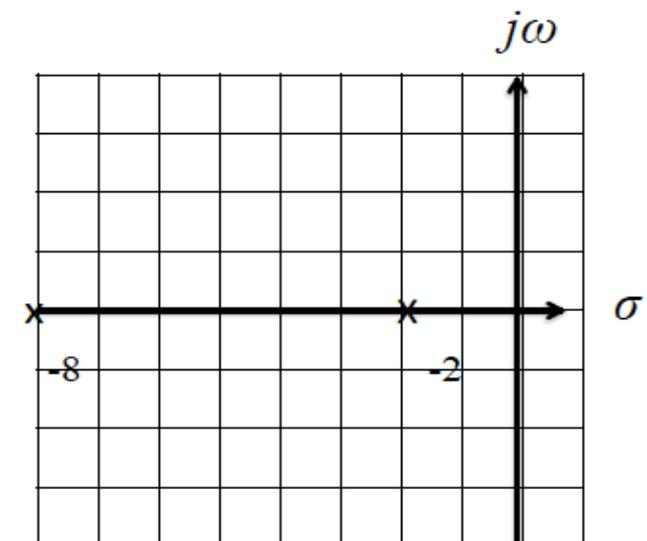


Figure 4.14

Find $c(t)$:

Laplace transform of the step response is: $C(s) = \frac{1}{s} \cdot \frac{16}{s^2 + 10s + 16} = \frac{16}{s(s+2)(s+8)}$

Expanding the equation into partial fraction,

$$\frac{16}{s(s+2)(s+8)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+8} \quad \rightarrow \quad A=1; B=-1.333; C=0.333$$

Applying the inverse Laplace transform:

$$c(t) = 1 - 1.333e^{-2t} + 0.333e^{-8t}$$

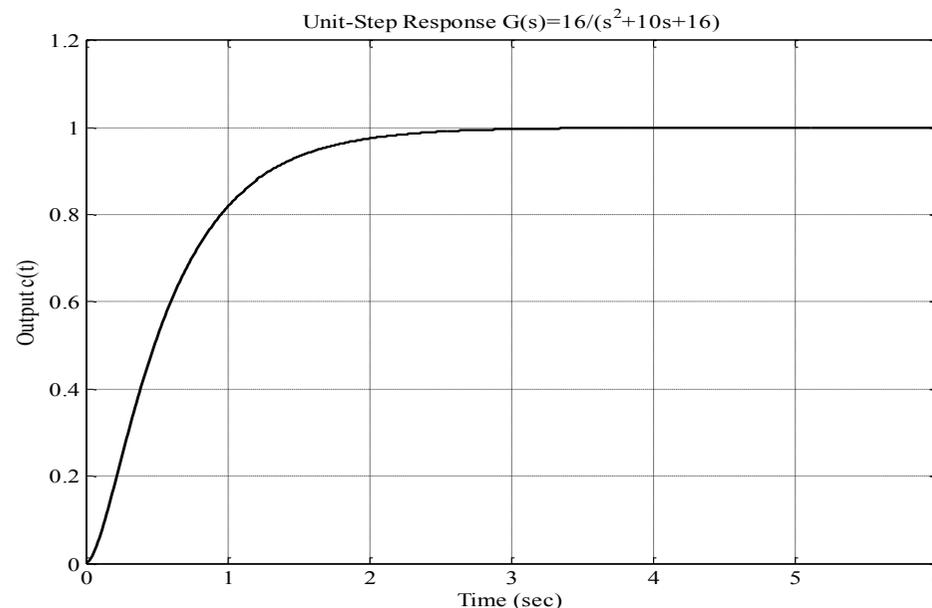


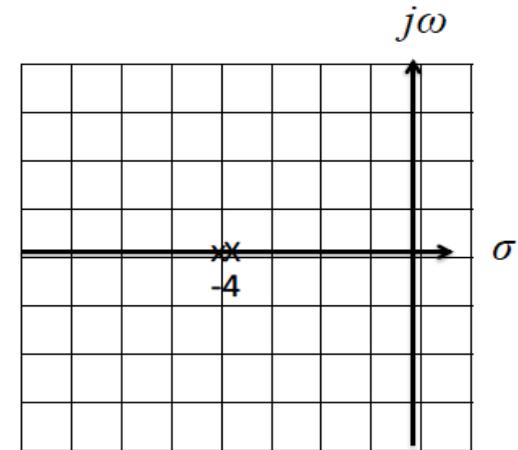
Figure 4.15 Plot $c(t)$ versus t , Over-damped response

b) Critically damped response

Transfer function:
$$\frac{C(s)}{R(s)} = \frac{16}{s^2 + 8s + 16}$$

Poles location on s-plane:

$$\frac{C(s)}{R(s)} = \frac{16}{(s+4)(s+4)} \rightarrow \text{poles are at } s=-4, s = -4$$



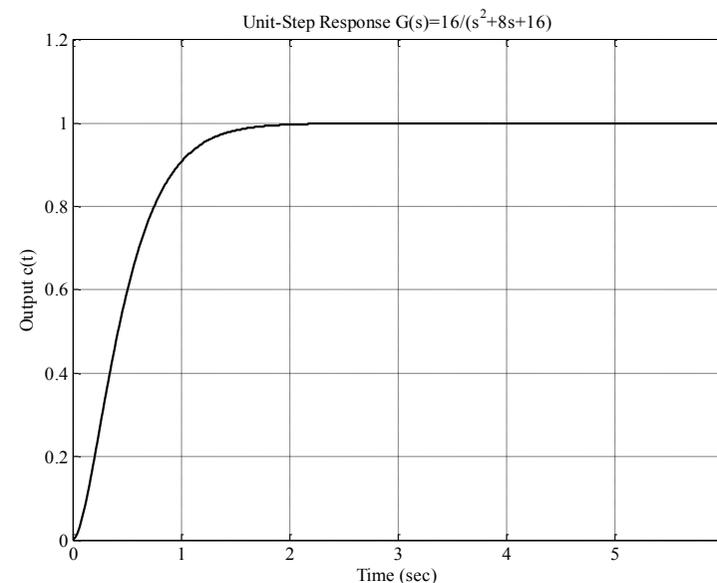
Expanding the equation into partial fraction:

$$\frac{16}{s(s+4)(s+4)} = \frac{1}{s} - \frac{4}{(s+4)^2} - \frac{1}{s+4}$$

Applying the inverse Laplace transform:

$$c(t) = 1 - 4te^{-4t} - 1e^{-4t}$$

Figure 4.16 Critically Damped Response



c) Under-damped response

Transfer function:
$$\frac{C(s)}{R(s)} = \frac{16}{s^2 + 4s + 16}$$

Poles location on s-plane:

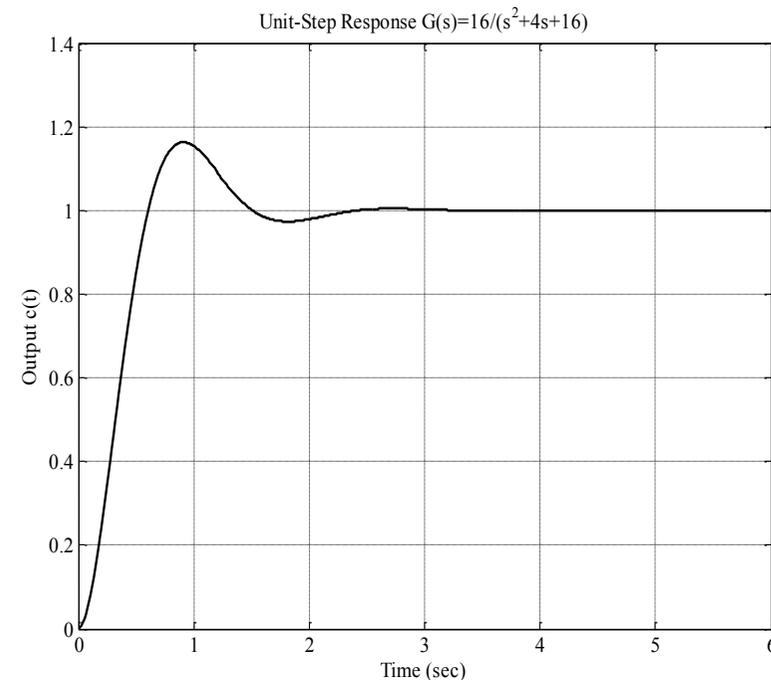
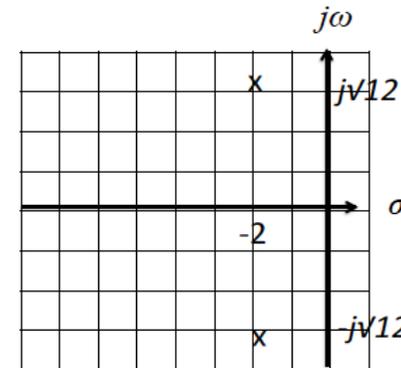
$$\frac{C(s)}{R(s)} = \frac{16}{(s + 2 + j\sqrt{12})(s + 2 - j\sqrt{12})}$$

→ poles are at $s = -2 - j\sqrt{12}$, $s = -2 + j\sqrt{12}$

Expanding the equation into partial fraction and apply the inverse Laplace transform, find $c(t)$

$$c(t) = 1 - e^{-2t} \left(\cos \sqrt{12}t + \frac{\sqrt{12}}{12} \sin \sqrt{12}t \right)$$

Figure 4.17 Underdamped response plot



d) Undamped response

Transfer function:
$$\frac{C(s)}{R(s)} = \frac{16}{s^2 + 16}$$

Poles location on s-plane:

$$\frac{C(s)}{R(s)} = \frac{16}{(s + j4)(s - j4)} \quad \rightarrow \text{poles are at } s = +j\sqrt{4}, s = -j\sqrt{4}$$

Expanding the equation into partial fraction and apply the inverse Laplace transform,

$$c(t) = 1 - \cos 4t$$

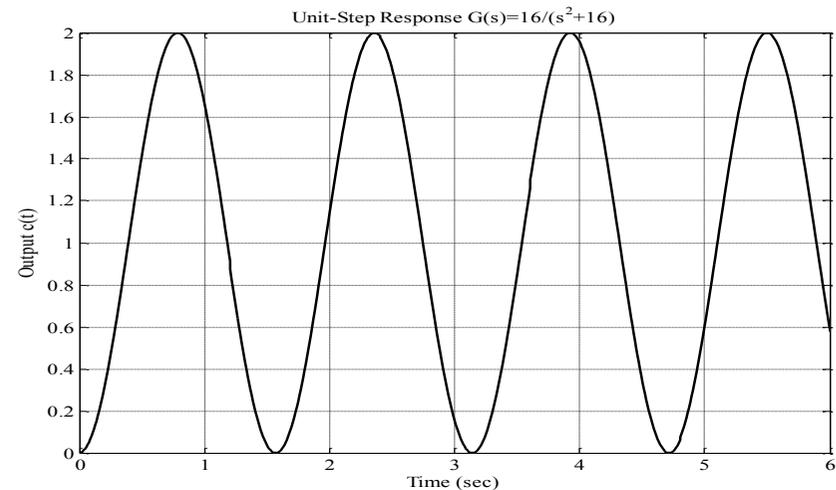
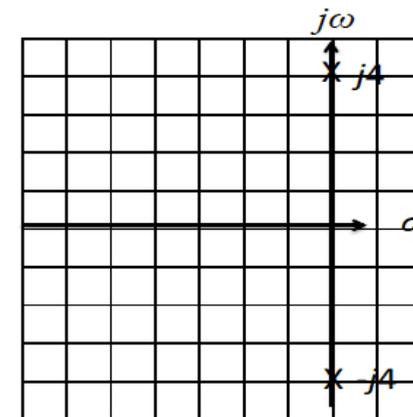
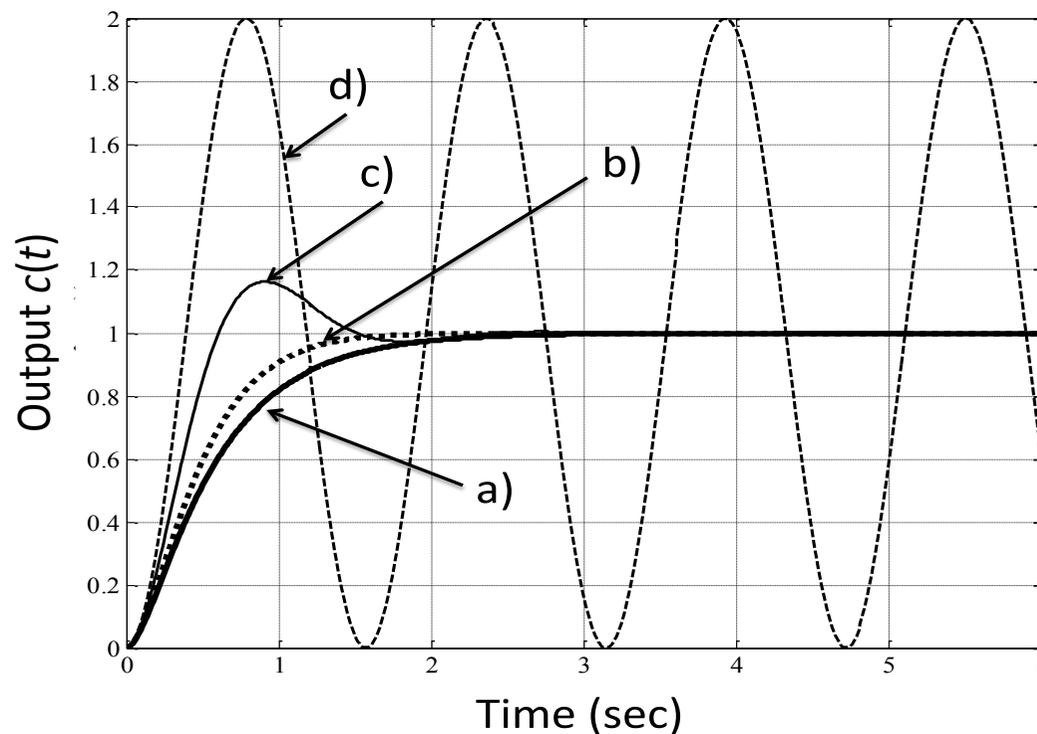


Figure 4.18 Undamped Response

For comparison, the transient response specifications for the 4 cases are presented in Figure 4.19



- a) Over-damped
- b) Critically-damped
- c) Under-damped
- d) Un-damped

Figure 4.19 Responses type

4.3.2 Under-Damped Response

The general transfer function of 2nd order system:

1. Natural frequency. ω_n
Frequency of oscillation of the system without damping
2. Damping Ratio. ζ

$$\zeta = \frac{\text{exponential decay frequency}}{\text{natural frequency (rad/sec)}}$$

Second-order system can be transformed to show the quantities of ζ and ω_n . Consider the general system

$$\frac{C(s)}{R(s)} = \frac{B}{s^2 + As + B}$$

so transfer function response without damping or undamped response

$$\frac{C(s)}{R(s)} = \frac{B}{s^2 + B} = \frac{\omega^2}{s^2 + \omega^2}$$

For poles purely imaginary, $A = 0$, poles on the $j\omega$ axis, $\omega_n = \pm j\sqrt{B}$; $B = \omega_n^2$

hence
$$\frac{C(s)}{R(s)} = \frac{B}{s^2 + B} = \frac{\omega^2}{s^2 + \omega^2}$$

For under-damped system, poles have real part, $= -A/2$

$$\zeta = \frac{\text{exponential decay frequency}}{\text{natural frequency (rad/sec)}} = \frac{|\sigma|}{\omega_n} = \frac{A/2}{\omega_n}$$

$$a = 2\zeta\omega_n$$

General second-order system transfer function:

$$\frac{C(s)}{R(s)} = G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

If the gain K is set to 1, the system can be described in Figures 20 and 21.

Poles:
$$S_{1,2} = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$$

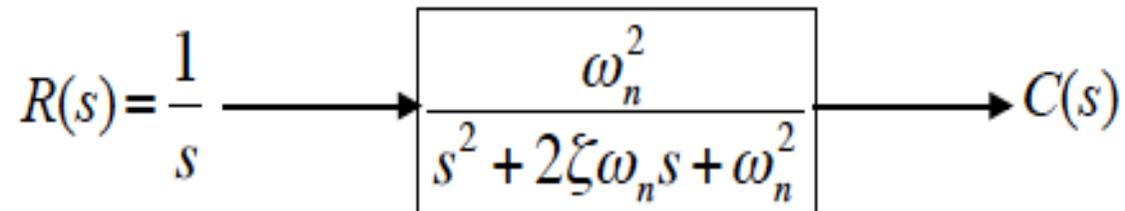


Figure 4.20 Closed-loop transfer function 2nd order system

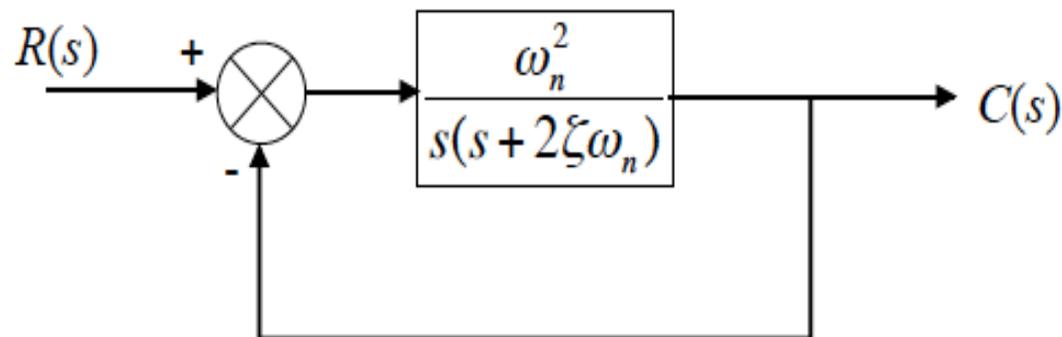


Figure 21 2nd order system

Transfer function from sub-topic 4.3.1

Find the values of ω_n , ζ . State the kind of response expected

$$\text{a) } \frac{C(s)}{R(s)} = \frac{16}{s^2 + 10s + 16}$$

$$\omega_n^2 = 16, \quad \omega_n = 4$$

$$2\zeta\omega_n = 10, \quad \zeta = 1.25$$

Find poles location ?

$$\text{b) } \frac{C(s)}{R(s)} = \frac{16}{s^2 + 4s + 16}$$

$$\omega_n^2 = 16, \quad \omega_n = 4$$

$$2\zeta\omega_n = 4, \quad \zeta = 0.5$$

Find poles location ?

$$\text{c) } \frac{C(s)}{R(s)} = \frac{16}{s^2 + 8s + 16}$$

$$\omega_n^2 = 16, \quad \omega_n = 4$$

$$2\zeta\omega_n = 8, \quad \zeta = 1$$

Find poles location ?

$$\text{d) } \frac{C(s)}{R(s)} = \frac{16}{s^2 + 16}$$

$$\omega_n^2 = 16, \quad \omega_n = 4$$

$$2\zeta\omega_n = 0, \quad \zeta = 0$$

Find poles location ?

Table 4.1 Damping ratio, poles location and response specifications

| ξ | Poles on s-plane | Response Specifications |
|---------------|--|----------------------------|
| $\xi = 0$ | Two poles at imaginary axis $s_{1,2} = \pm j\omega_n$ | Un-damped response |
| $0 < \xi < 1$ | Two poles at complex s-plane $s_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$ | Under-damped response |
| $\xi = 1$ | Two poles at real and same point $s_{1,2} = -\xi\omega_n$ | Critically-damped response |
| $\xi > 1$ | Two poles at real $s_{1,2} = -\xi\omega_n \mp \omega_n\sqrt{\xi^2-1}$ | Over-damped response |

4.3.3 Time Response Specifications

To derive time response specifications:

$$\frac{C(s)}{R(s)} = G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Used PDE and unit step ($1/s$) test input signal: $R(s) = \frac{1}{s}$

$$C(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Assume that $\zeta < 1$, case underdamped, obtain K_1 , K_2 and K_3 after that taking the Laplace transform; produces:

$$c(t) = 1 - e^{-\zeta\omega_n t} \left(\cos \omega_n \sqrt{1 - \zeta^2} t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_n \sqrt{1 - \zeta^2} t \right)$$

a) Rise time, T_r

Time required for waveform to go from 0.1 (10%) of final value to 0.9 (90%) of the final value

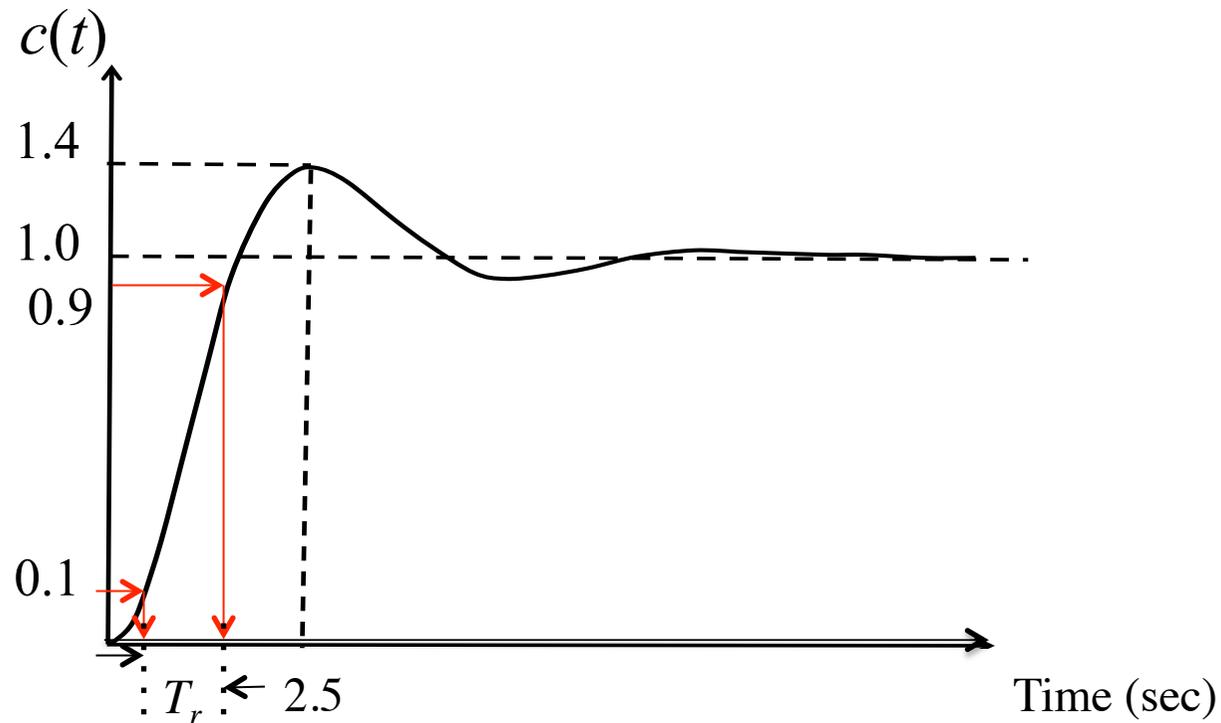


Figure 4.22

b) Peak Time, T_p

Time required to reach the first/maximum peak

Evaluation of T_p

- Differentiating $c(t)$
- Find the first zero after $t=0$
- Assuming zero initial conditions

$$\dot{c}(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2} t\right)$$

$$\dot{c}(t) = 0 \quad \text{if} \quad \omega_n \sqrt{1-\zeta^2} t = n\pi \rightarrow$$

$$\rightarrow t = T_p = \frac{n\pi}{\omega_n \sqrt{1-\zeta^2}} \quad \text{The 1st peak } n = 1$$

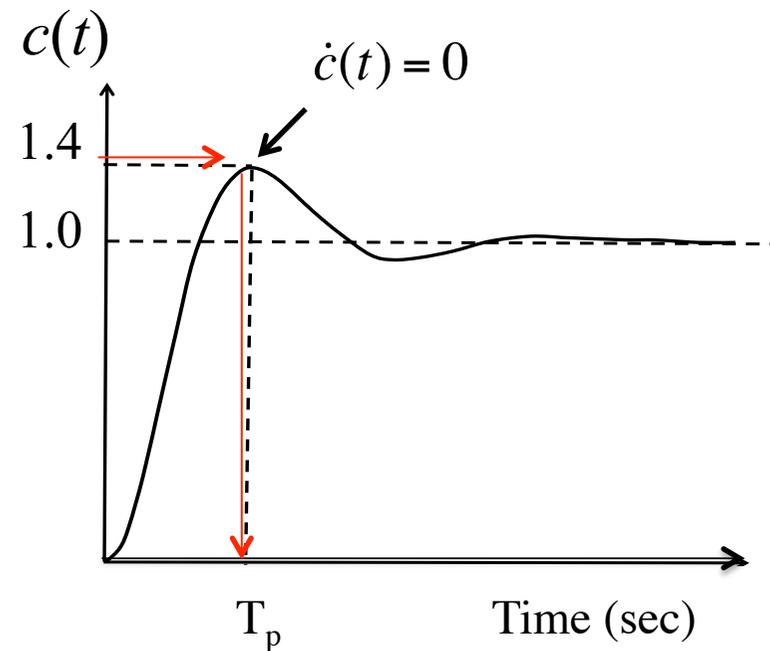


Figure 4.23

c) Percentage overshoot: %OS

The amount that waveform overshoots from the steady-state or final value

C_{\max} : from $c(t)$ at T_p

$$C_{\max} = c(T_p) = 1 - e^{-\zeta\omega_n T_p} \cos(\omega_n \sqrt{1-\zeta^2} t - \phi)$$

$$C_{\max} = c(T_p) = 1 - e^{-\zeta\pi / \sqrt{1-\zeta^2}}$$

$$\%OS = \frac{C_{\max} - C_{\text{final}}}{C_{\text{final}}} \times 100\%$$

\therefore if $C_{\text{final}} = 1$

$$\%OS = e^{-\zeta\pi / \sqrt{1-\zeta^2}} \times 100$$

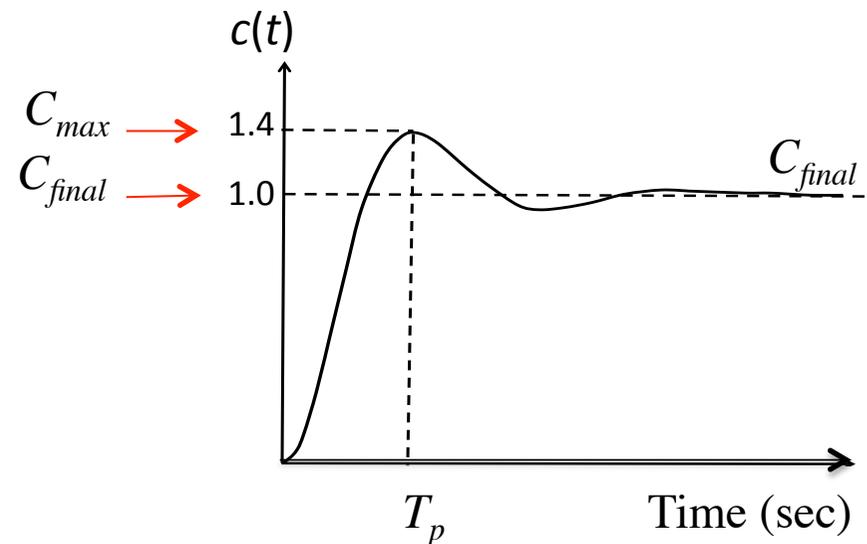


Figure 4.24

d) Settling Time, T_s

The time $c(t)$ reaches and stay within $\pm 2\%$ ($\pm 4\%$) of the steady state value C_{final}

$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos(\omega_n \sqrt{1-\zeta^2} t - \phi)$$

$$\cos(\omega_n \sqrt{1-\zeta^2} t - \phi) = 1, \quad t = T_s$$

$$c(t) - C_{final} = 0.02$$

$$\Rightarrow \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n T_s} = 0.02$$

$$\Rightarrow T_s = \frac{-\ln(0.02\sqrt{1-\zeta^2})}{\zeta\omega_n} \approx \frac{4}{\zeta\omega_n}$$

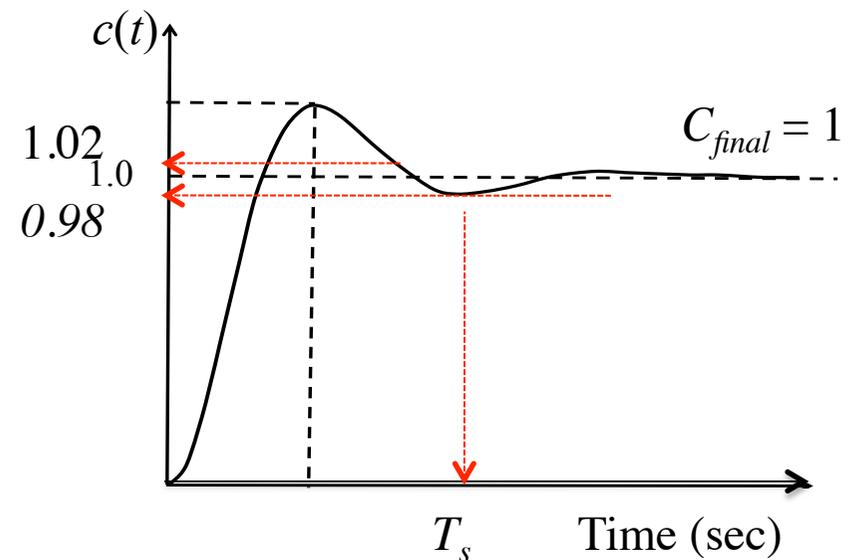


Figure 4.25

Example 4.6: Find T_p , %OS and T_s

$$\frac{C(s)}{R(s)} = \frac{16}{s^2 + 4s + 16}$$

$$\frac{C(s)}{R(s)} = \frac{16}{(s + 2 + j\sqrt{12})(s + 2 - j\sqrt{12})}$$

$$T_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} = \frac{\pi}{\sqrt{12}} s$$

$$T_s = \frac{4}{\zeta \omega_n} = \frac{4}{2} = 2s$$

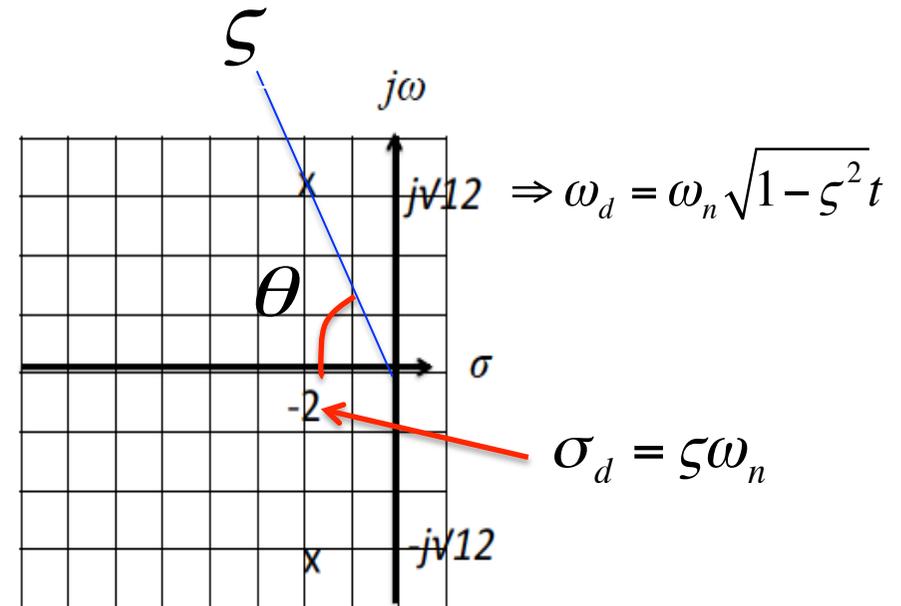


Figure 4.26

Find θ

$$\zeta = \cos \theta$$

$$\%OS = e^{-\zeta\pi / \sqrt{1 - \zeta^2}} \times 100$$

Example 4.7

Find the transfer function of the following 2nd order system

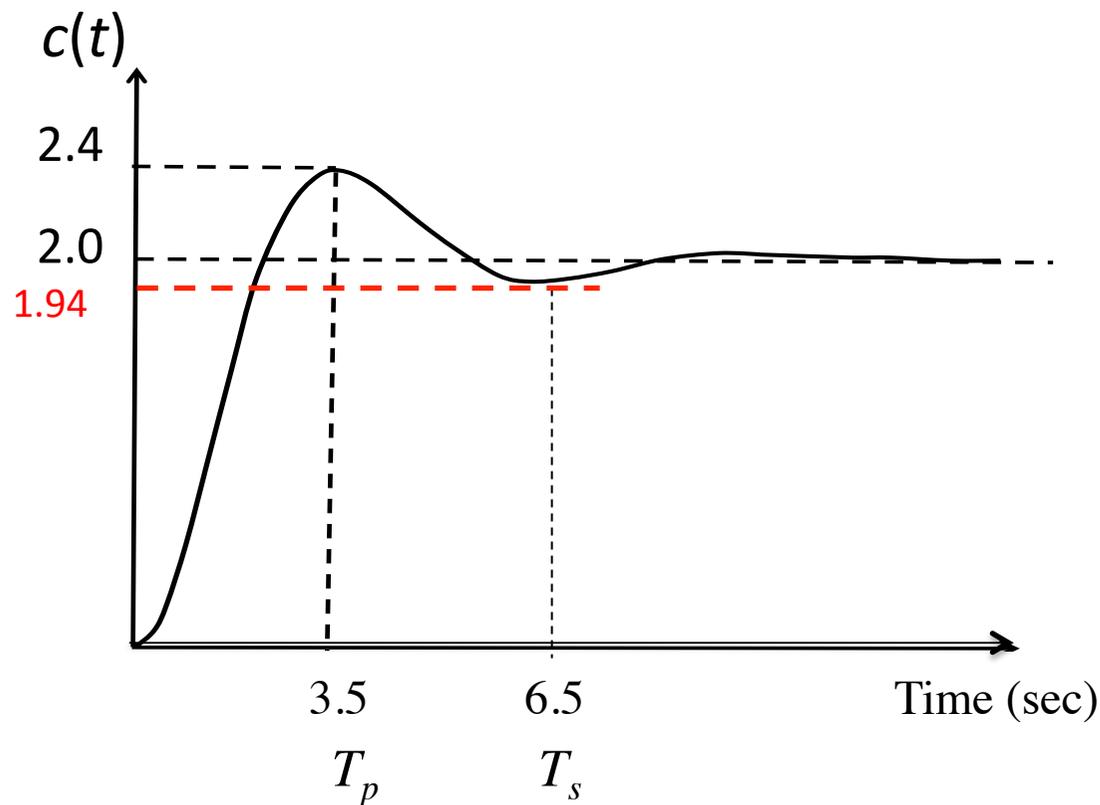


Figure 4.27

4.4 Steady State Error

Steady state error (e_{ss}) is the difference between input and output for a prescribed test input as $t \rightarrow \infty$.

$$e_{ss}(\infty) = \left| r(t) - c(t) \right|_{t \rightarrow \infty}$$

Case I: in terms of $T(s)$ (Closed-loop TF)

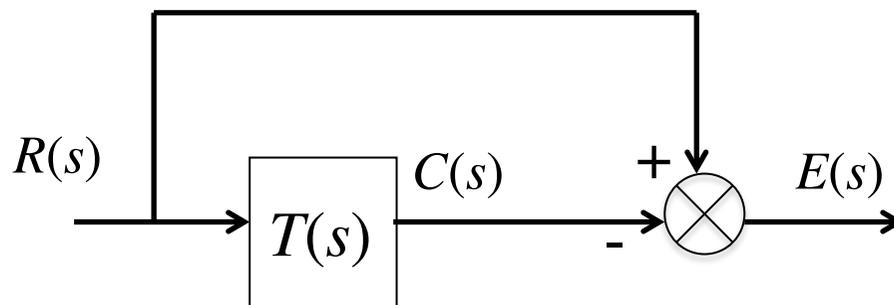


Figure 4.28

$$E(s) = \text{input} - \text{output}$$

$$= R(s) - C(s)$$

$$C(s) = R(s) T(s)$$

$$E(s) = R(s) [1 - T(s)]$$

Using Final Value Theorem:

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s)$$

$$e_{ss} = \lim_{s \rightarrow 0} sR(s)[1 - T(s)]$$

Problem 4.8: Find the steady state error for a unit step input

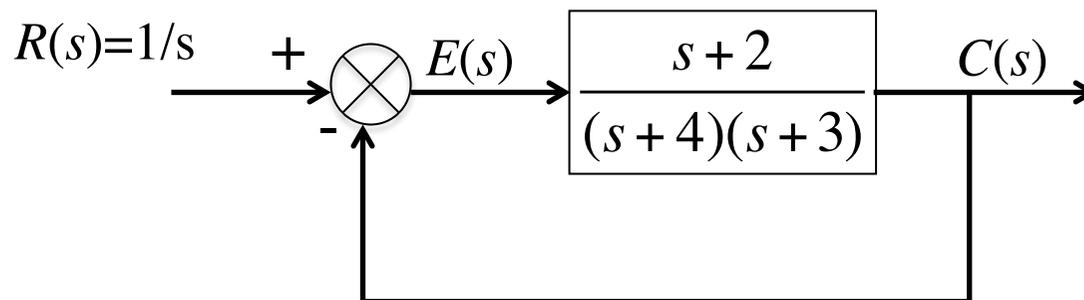


Figure 4.29

$$T(s) = \frac{s+2}{(s+4)(s+3) - (s+2)}$$

$$e_{ss} = \lim_{s \rightarrow 0} s \cdot \frac{1}{s} \cdot \left[1 - \frac{s+2}{(s+4)(s+3) - (s+2)} \right] = 1 - \frac{2}{10} = 0.8$$

Case II: in terms of $G(s)$ (based on unity feedback system)

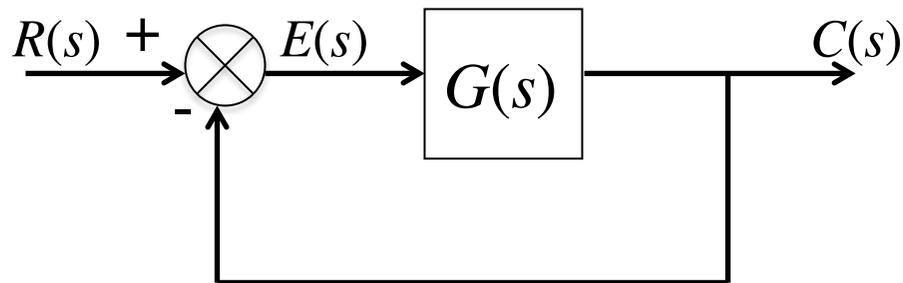


Figure 4.30

Forward TF: $G(s)$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

$$E(s) = \text{input} - \text{output}$$

$$= R(s) - C(s)$$

$$= R(s) - \frac{G(s)}{1 + G(s)} \cdot R(s)$$

$$= R(s) \left[1 - \frac{G(s)}{1 + G(s)} \right]$$

$$= R(s) \left[\frac{1 + G(s) - G(s)}{1 + G(s)} \right]$$

$$= \frac{1}{1 + G(s)} \cdot R(s)$$

Using Final Value Theorem:

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s)$$

$$e_{ss} = \lim_{s \rightarrow 0} \frac{sR(s)}{1 + G(s)}$$

Problem 4.9: Find the steady state error for a unit step input

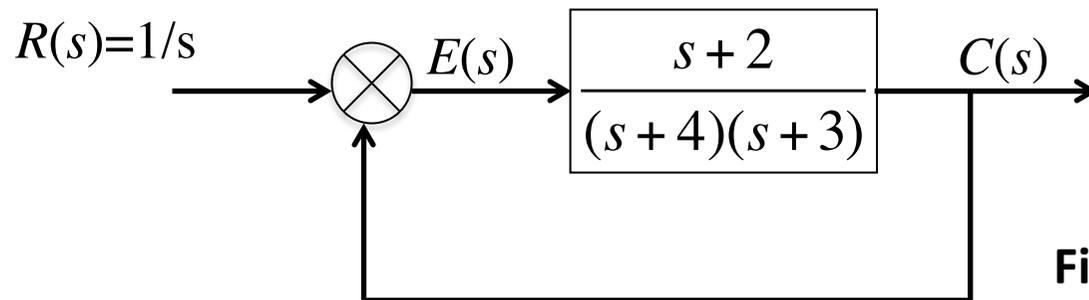


Figure 4.31

$$e_{ss} = \lim_{s \rightarrow 0} \frac{s \cdot \frac{1}{s}}{1 + \frac{s+2}{(s+4)(s+3)}} = \frac{1}{1 + \frac{2}{(4)(3)}} = 0.86$$

4.5 Stability Analysis in Time Domain

4.5.1 Relative stability (stability based on poles location)

- The total response of a system is the sum of the forced and natural responses:

$$c(t) = c_{\text{forced}} + c_{\text{natural}}(t)$$

$$c(t) = K_1 + K_2 e^{-\alpha_1 t} + K_3 e^{-\alpha_2 t} + K_4 e^{-\alpha_3 t} + \dots$$



- There are three cases about the system stability:
 - ✓ A system is **stable** if the natural response approaches zero as time approaches infinity.
 - ✓ A system is **unstable** if the natural response approaches infinity as time approaches infinity.
 - ✓ A system is **marginally stable** if the natural response neither decays nor grows but remains constant or oscillates.

a) A Stable System

- Poles in the left half-plane (LHP) yield either pure exponential decay or damped sinusoidal natural responses.
- Thus, if the closed-loop system poles are in the LHP, the system is stable.

b) Unstable System

- Poles in the right half-plane (RHP) yield either pure exponentially increasing or exponentially increasing sinusoidal natural responses.
- These natural responses approach infinity as time approaches infinity
- Thus, if the closed-loop system poles are in the RHP or at least one pole, the system is unstable.

c) Marginally Stable System

- A system that has imaginary axis poles yields pure sinusoidal oscillations as a natural response.
- Thus, marginally stable systems have closed-loop transfer functions with only imaginary axis poles or/and poles in the LHP.

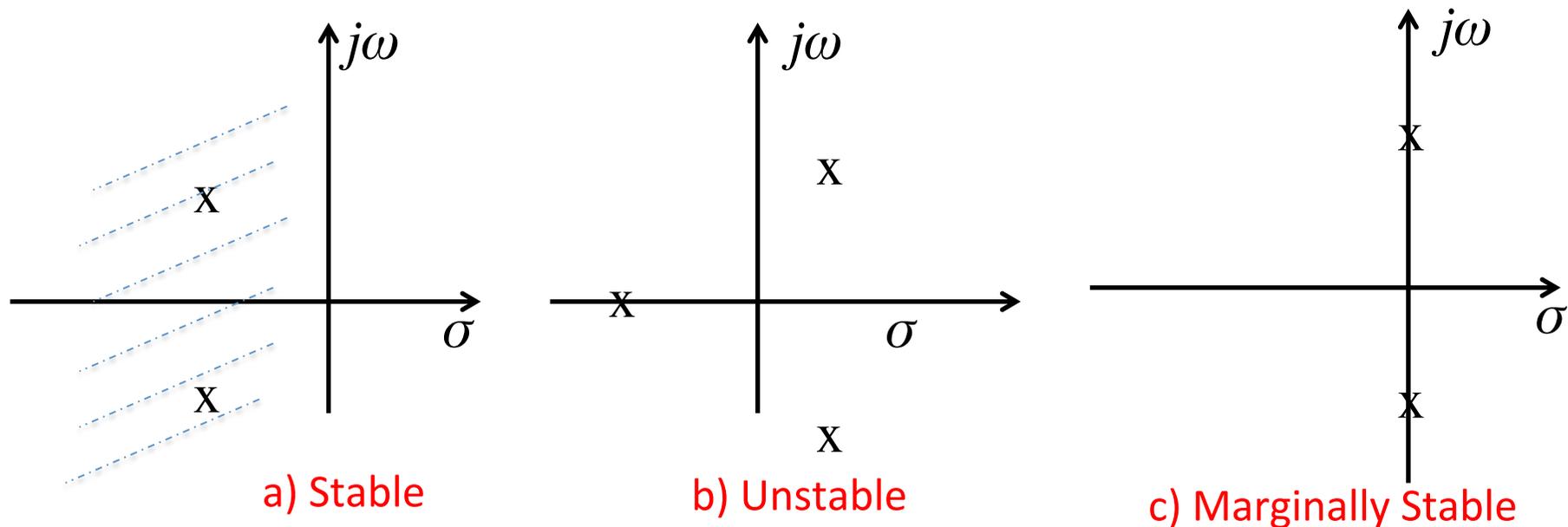


Figure 4.32

4.5.2 Methods to Test Stability: Routh Hurwitz Criterium

- ❑ RH criterion is a method that yields stability information without the need to identify for the closed-loop system poles.
- ❑ This method can tell how many closed-loop system poles are in the LHP (left-hand-poles), RHP (right-hand-poles), or on the imaginary axis, but we cannot find their locations.
- ❑ There are 2 steps in RH criterion approaches:
 - ✓ Generate the Routh Table
 - ✓ Interpret the table for system stability
- ❑ Interpreting the RH Table:
 - ✓ The RH criterion declares that *the number of roots of the polynomial that are in the RHP is equal to the number of sign changes in the first column.*
 - ✓ A system is stable if all the CL poles lie on the LHP. Thus, a system is *stable* if there is *no sign change* in the first column of the RH table.

Generate Routh Table

- Consider the closed-loop transfer function is $T(s)$.
- The denominator of the transfer function relates to the poles location of the system.

$$T(s) = \frac{C(s)}{R(s)} = \frac{N(s)}{a_6s^6 + a_5s^5 + a_4s^4 + a_3s^3 + a_2s^2 + a_1s^1 + a_0s^0}$$

Steps of generating Routh Table from the denominator (Table 4.2)

$$a_6s^6 + a_5s^5 + a_4s^4 + a_3s^3 + a_2s^2 + a_1s^1 + a_0s^0$$

- Labeling the rows from s^6 to s^0
- Fill in coefficients a_6 to a_0 in the 1st and 2nd rows alternately

Table 4.2

| | | | | |
|-------|-------|-------|-------|-------|
| s^6 | a_6 | a_4 | a_2 | a_0 |
| s^5 | a_5 | a_3 | a_1 | 0 |
| s^4 | | | | |
| s^3 | | | | |
| s^2 | | | | |
| s^1 | | | | |
| s^0 | | | | |

- The remaining entries 3rd row until the end: s^4 to s^0 :
- Each entry is a -ve determinant of entries in the previous two rows divided by the entry in the first column directly above the calculated row.
- The left-hand column of the determinant is always the first column of the previous two rows, and the right-hand column is the elements of the column above and to the right

Table 4.3

| | | | | | |
|-------|---|---|---|---|---|
| s^6 | a_6 | a_4 | a_2 | a_0 | 0 |
| s^5 | a_5 | a_3 | a_1 | 0 | 0 |
| s^4 | $b_1 = -\frac{\begin{vmatrix} a_6 & a_4 \\ a_5 & a_3 \end{vmatrix}}{a_5}$ | $b_2 = -\frac{\begin{vmatrix} a_6 & a_2 \\ a_5 & a_1 \end{vmatrix}}{a_5}$ | $b_3 = -\frac{\begin{vmatrix} a_6 & a_0 \\ a_5 & 0 \end{vmatrix}}{a_5}$ | $b_4 = -\frac{\begin{vmatrix} a_6 & 0 \\ a_5 & 0 \end{vmatrix}}{a_5}$ | |
| s^3 | $c_1 = -\frac{\begin{vmatrix} a_5 & a_3 \\ b_1 & b_2 \end{vmatrix}}{b_1}$ | $c_2 = -\frac{\begin{vmatrix} a_5 & a_1 \\ b_1 & b_3 \end{vmatrix}}{b_1}$ | ? | ? | |
| s^2 | $d_1 = -\frac{\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}}{c_1}$ | ? | ? | ? | |
| s^1 | $e_1 = ??$ | ? | ? | ? | ? |
| s^0 | $f_1 = ?$ | ? | ? | ? | ? |

Example 4.10: Determine the stability of the following system Figure 4.33 using RH method.

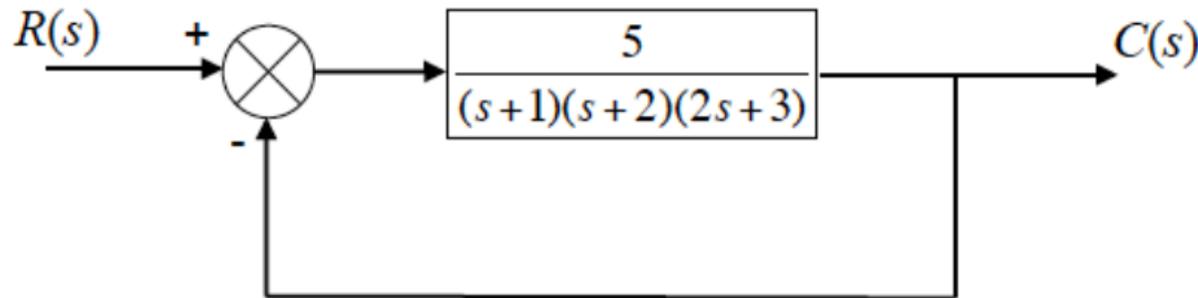


Figure 4.33

Solution:

Closed-loop transfer function

$$T(s) = \frac{(s+1)(s+2)(2s+3)}{1 + \frac{5}{(s+1)(s+2)(2s+3)}} = \frac{5}{2s^3 + 9s^2 + 13s^1 + 11s^0}$$

Complete the RH Table 4.4

Table 4.4

| | | | |
|-------|---|----|--|
| s^3 | 2 | 13 | 0 |
| s^2 | 9 | 11 | 0 |
| s^1 | $\frac{- \begin{vmatrix} 2 & 13 \\ 9 & 11 \end{vmatrix}}{9} = \frac{95}{9}$ | | $\frac{- \begin{vmatrix} 2 & 0 \\ 9 & 0 \end{vmatrix}}{9} = 0$ |
| s^0 | $\frac{- \begin{vmatrix} 3 & 2 \\ 95/9 & 0 \end{vmatrix}}{95/9} = 2$ | | 0 |



| | |
|-------|-----|
| s^3 | +ve |
| s^2 | +ve |
| s^1 | +ve |
| s^0 | +ve |

↑
1st column

↑
Sign of 1st column

- ✓ No sign change in the first column of the RH table
- ✓ All poles at the left s-plane (LHP)
- ✓ **System is stable**

Example 4.11: Determine the stability of the following closed-loop system using RH method.

$$T(s) = \frac{45}{s^3 + 5s^2 + 8s + 50}$$

Solution:

Table 4.5

| | | | |
|-------|---|--------|---|
| s^3 | 1 | 8 | 0 |
| s^2 | 5, 1 | 50, 10 | 0 |
| s^1 | $-\frac{\begin{vmatrix} 1 & 8 \\ 1 & 10 \end{vmatrix}}{1} = -2$ | | $-\frac{\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$ |
| s^0 | $-\frac{\begin{vmatrix} 1 & 10 \\ -2 & 0 \end{vmatrix}}{-2} = 10$ | | 0 |

↑
1st column



| | |
|-------|-----|
| s^3 | +ve |
| s^2 | +ve |
| s^1 | -ve |
| s^0 | +ve |

↑
Sign changes



- ✓ 2 times sign changes in the first column of the RH table, from:
 - $s^2 \rightarrow s^1$ (+ve to -ve)
 - $s^1 \rightarrow s^0$ (-ve to +ve)
- ✓ 2 poles at the right s-plane (RHP) and 2 at the LHP
- ✓ **System is unstable**

RH with entire Row is Zero

Example 4.12: Consider the following system TF

$$T(s) = \frac{1}{2s^5 + 6s^4 + 3s^3 + 27s^2 + 6s + 18}$$

Complete RH:

Table 4.6

| | | | | |
|-------|------|-------|-------|---|
| s^5 | 2 | 3 | 6 | 0 |
| s^4 | 6, 2 | 27, 9 | 18, 6 | 0 |
| s^3 | 0 | 0 | 0 | |
| s^2 | | | | |
| s^1 | | | | |
| s^0 | | | | |

Interpreting Row of Zero (ROZ):

Return to the row immediately above the ROZ and form a polynomial using the entries in that row as coefficients.

$$P(s) = 6s^4 + 27s^2 + 18$$

- Differentiate the polynomial with respect to s .

$$dP(s)/ds = 24s^3 + 54s$$
- Use the coefficients to replace the ROZ.
- Then complete the RH table from s^3 .
- When there is ROZ, poles might be located on jw -axis (in this case – 4 poles since ROZ is at s^3).
- There is no sign change in the first column after the ROZ. Hence, there are no RHP poles. The system is marginally stable

Table 4.6 cont.

| | | | | |
|-------|----------|----------|-------|---|
| s^5 | 2 | 3 | 6 | 0 |
| s^4 | 6, 2 | 27, 9 | 18, 6 | 0 |
| s^3 | 0, 24, 4 | 0, 54, 9 | 0 | |
| s^2 | 4.5 | 6 | 0 | |
| s^1 | 3.67 | 0 | | |
| s^0 | 6 | 0 | | |

Example 4.13: Consider the following system TF

$$T(s) = \frac{1}{s^5 + 3s^4 + 3s^3 + 9s^2 + 5s + 15}$$

Complete RH:

Table 4.7

| | | | | |
|-------|----------|----------|-------|---|
| s^5 | 1 | 3 | 5 | 0 |
| s^4 | 3, 1 | 9, 3 | 15, 5 | 0 |
| s^3 | 0, 12, 2 | 0, 18, 3 | 0 | |
| s^2 | 1.5 | 5 | 0 | |
| s^1 | -3.6 | 0 | | |
| s^0 | 5 | 0 | | |

Interpreting Row of Zero (ROZ):

Return to the row immediately above the ROZ and form a polynomial using the entries in that row as coefficients.

$$P(s) = 3s^4 + 9s^2 + 15$$

- Differentiate the polynomial with respect to s .

$$dP(s)/ds = 12s^3 + 18s$$
- Use the coefficients to replace the ROZ.
- Then complete the RH table frpm s^3 .
 When there is ROZ, poles might be located on $j\omega$ -axis since ROZ is at s^3 .
- There are 2 signs change in the first column after the ROZ. Hence, there are 2 RHP, and 2 LHP.
- **The system is unstable**

Check 1st column



| | | |
|-------|-----|-----|
| s^5 | +ve | |
| s^4 | +ve | |
| s^3 | +ve | |
| s^2 | +ve |) 1 |
| s^1 | -ve |) 2 |
| s^0 | +ve | |



Sign changes

RH-Zero only in the First Column

- For Zero only in the first column, an epsilon, (ε) is assigned to replace the zero.
- The value is allowed to approach zero from positive and negative. The signs of entries of the first column is then analyzed with the value, ε .

Example 4.13: Determine the number of RHP poles in the CLTF

$$T(s) = \frac{1}{s^5 + 3s^4 + 3s^3 + 9s^2 + 6s + 9}$$

Table 4.8

| | | | | |
|-------|---|---|---|---|
| s^5 | 1 | 3 | 6 | 0 |
| s^4 | 3 | 9 | 9 | 0 |
| s^3 | $0, \varepsilon$ | 3 | 0 | |
| s^2 | $-\frac{\begin{vmatrix} 3 & 9 \\ \varepsilon & 3 \end{vmatrix}}{\varepsilon} = -\frac{9-9\varepsilon}{\varepsilon}$ | 9 | | |
| s^1 | $-\frac{\begin{vmatrix} \varepsilon & 3 \\ \frac{9-9\varepsilon}{\varepsilon} & 9 \end{vmatrix}}{\frac{9-9\varepsilon}{\varepsilon}} = -\frac{9\varepsilon^2 + 27\varepsilon - 27}{9-9\varepsilon}$ | 0 | | |
| s^0 | 9 | 0 | | |



- 1st column
 ε is assigned to
 replace the
 zero.

Table 4.8 cont.

- Complete the table with positive or negative values of ε , and check the first column sign

| Label | Ist column | $+\varepsilon$ | $-\varepsilon$ |
|-------|---|----------------|----------------|
| s^5 | 1 | + | + |
| s^4 | 3 | + | + |
| s^3 | θ, ε | + | - |
| s^2 | $-\frac{\begin{vmatrix} 3 & 9 \\ \varepsilon & 3 \end{vmatrix}}{\varepsilon} = -\frac{9-9\varepsilon}{\varepsilon}$ | - | + |
| s^1 | $-\frac{\begin{vmatrix} \varepsilon & 3 \\ \frac{9-9\varepsilon}{\varepsilon} & 9 \end{vmatrix}}{\frac{9-9\varepsilon}{\varepsilon}} = -\frac{9\varepsilon^2 + 27\varepsilon - 27}{9-9\varepsilon}$ | + | - |
| s^0 | 9 | + | + |

signs change, 2
 RHP poles, the
 system is
 unstable.

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