SKMM 3033
Finite Element Method

Topic 7: Prismatic beam elements

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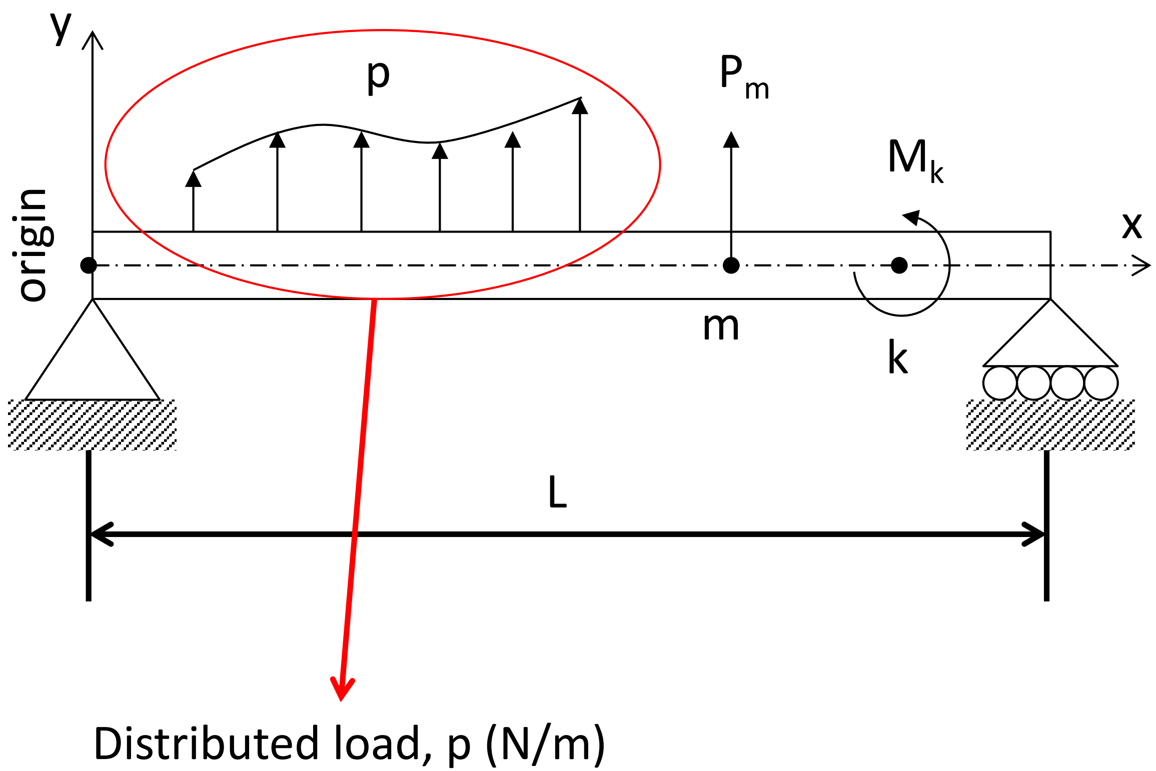
By the end of the notes:

The students are expected:

- To assemble the stiffness matrix and load vector for a beam element
- To assemble the global stiffness matrix for a beam structure

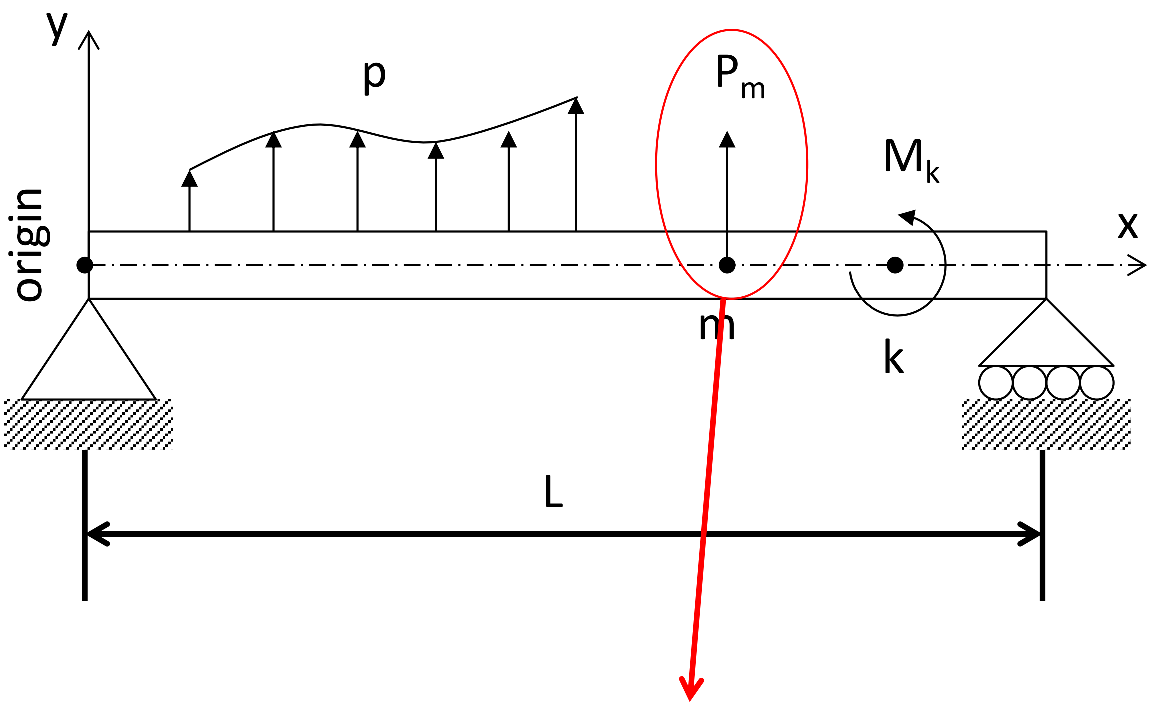
Beam Analysis



In FEM, we are interested mainly in:

1. **Loading condition**
2. Displacement, q
3. Stiffness matrix, k

Beam Analysis

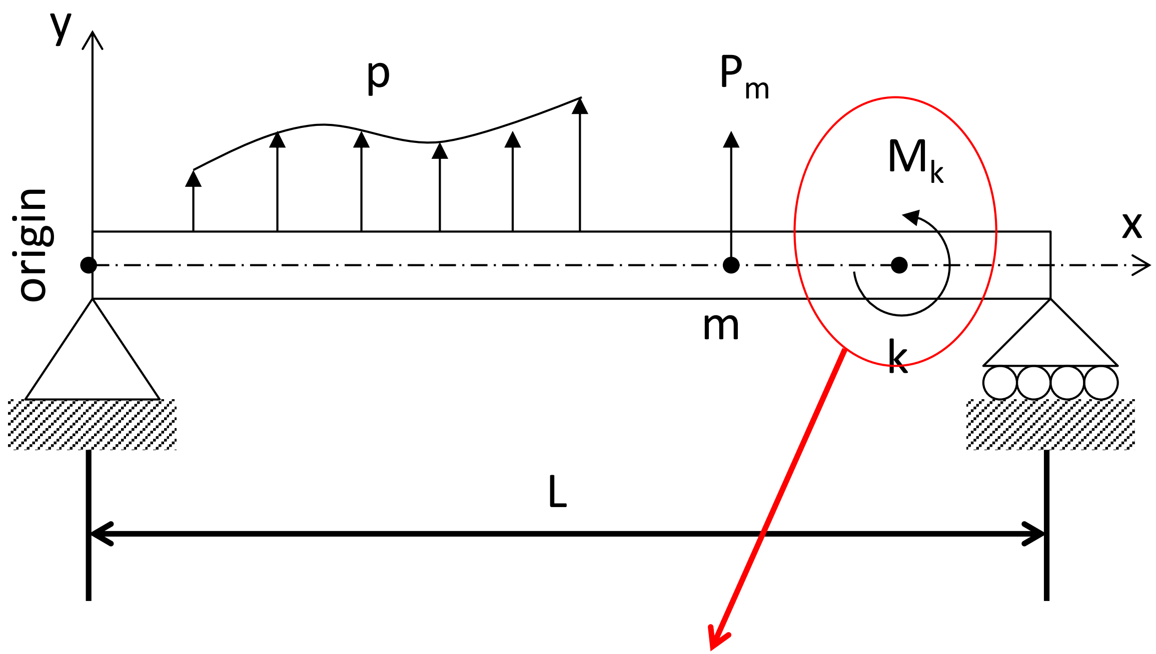


Concentrated load, P (N)

In FEM, we are interested mainly in:

1. **Loading condition**
2. Displacement, q
3. Stiffness matrix, k

Beam Analysis

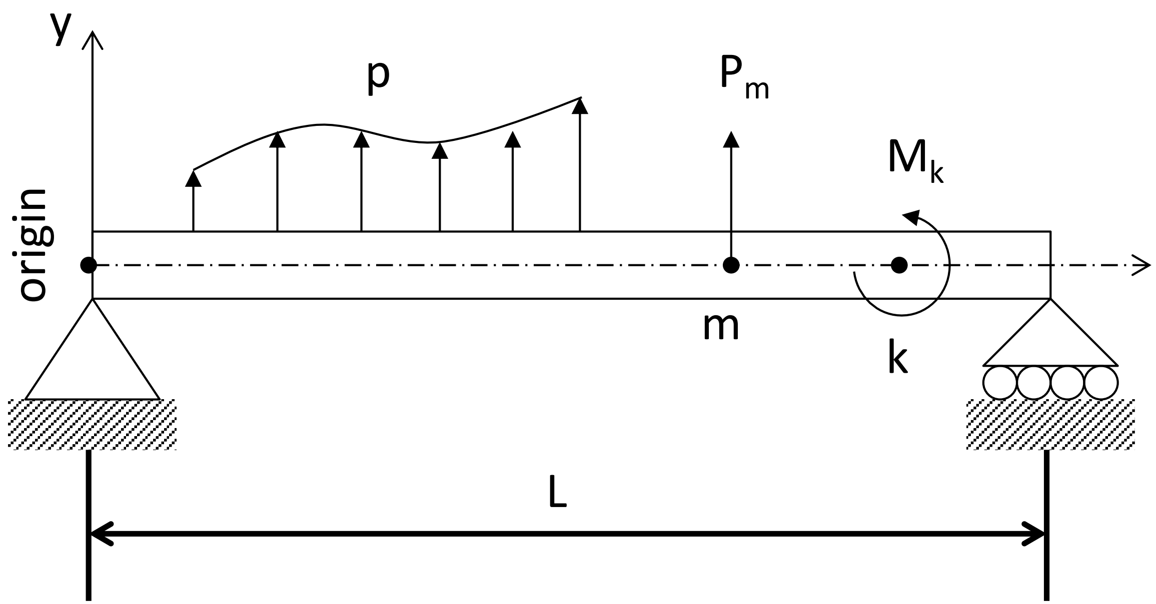


Concentrated moment, M (Nm)

In FEM, we are interested mainly in:

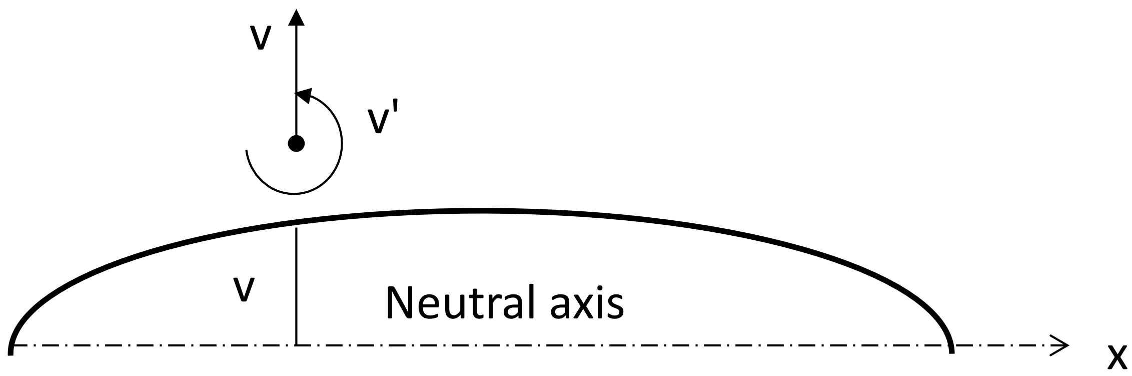
1. **Loading condition**
2. Displacement, q
3. Stiffness matrix, k

Beam Analysis



In FEM, we are interested mainly in:

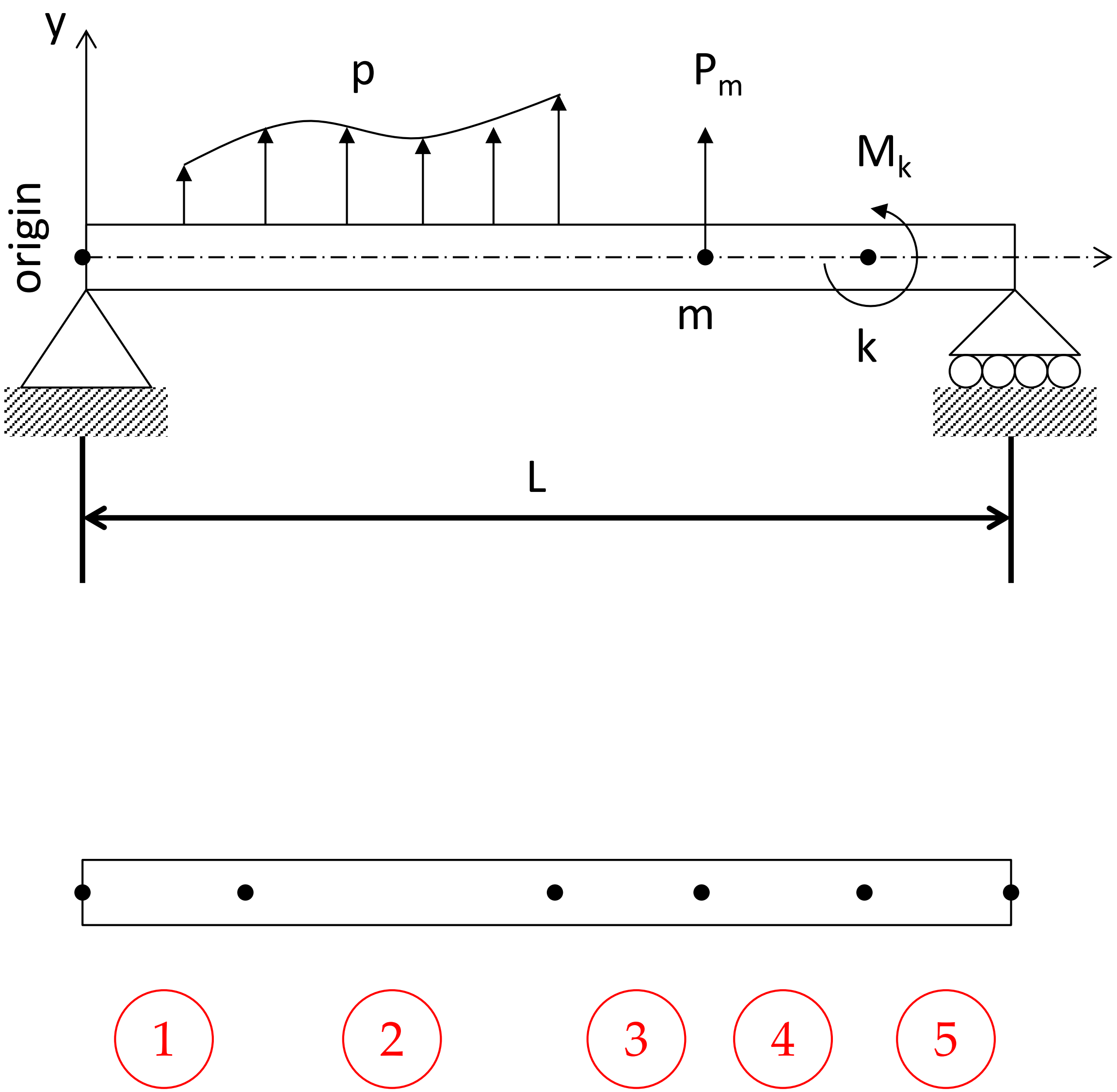
1. Loading condition
2. **Displacement, q**
3. Stiffness matrix, k



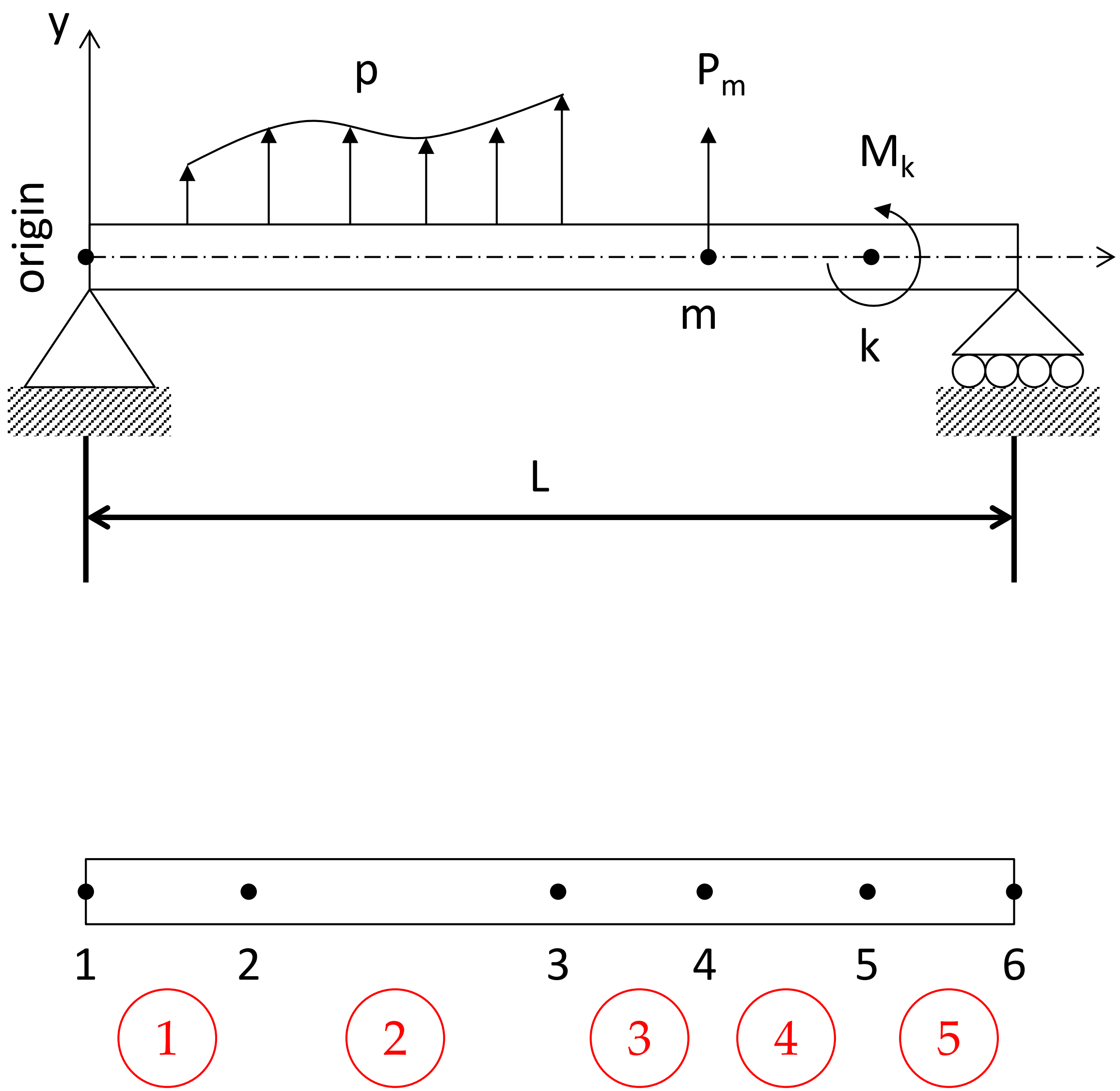
Deformation along neutral axis:

v=vertical deflection
v'=rotation or slope

FEA - Beams



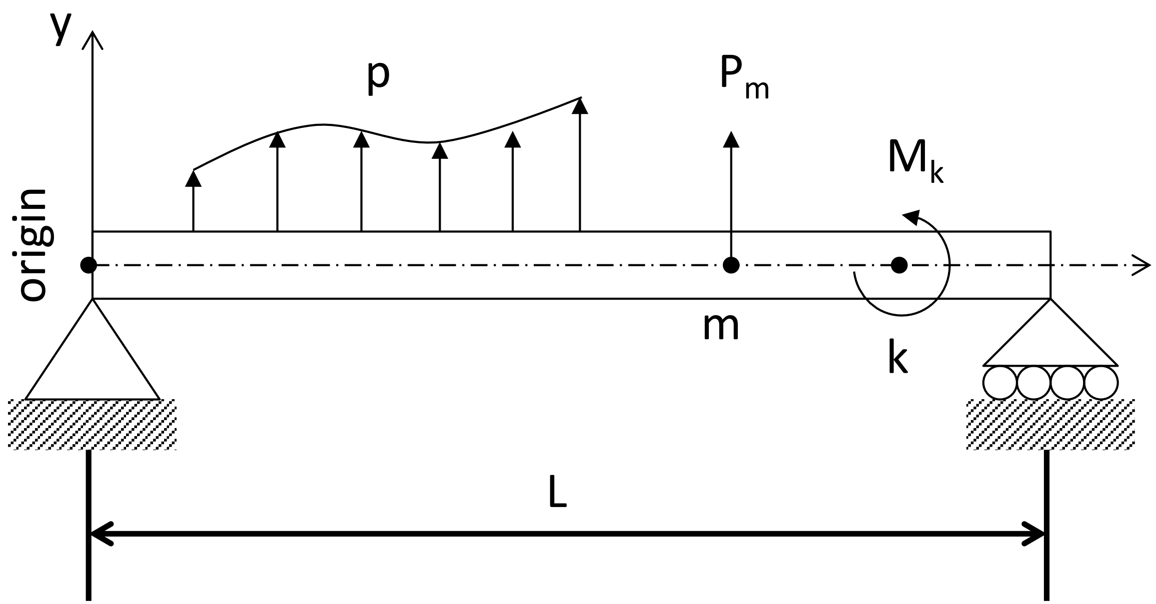
FEA - Beams

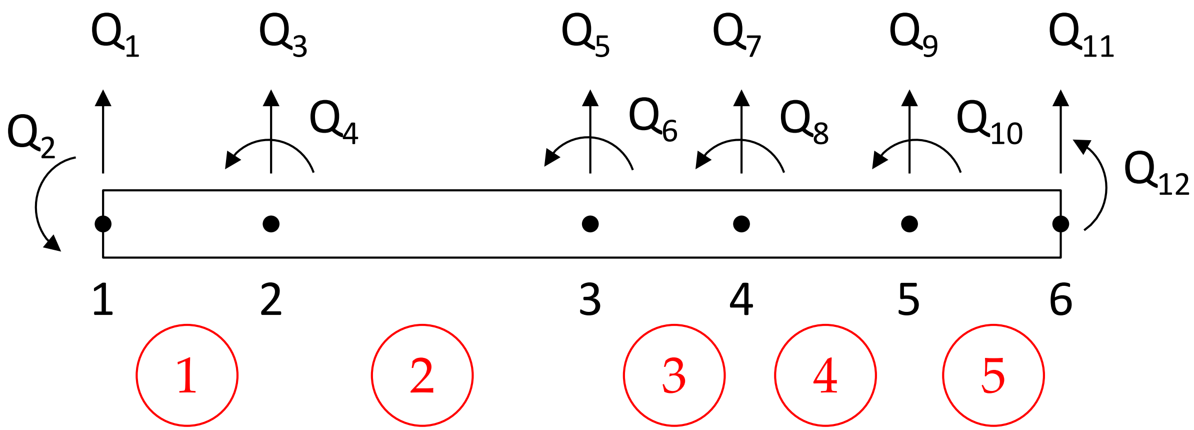


Element Connectivity Table

Element	Nodes	
1	1	2
2	2	3
3	3	4
4	4	5
5	5	6

FEA - Beams

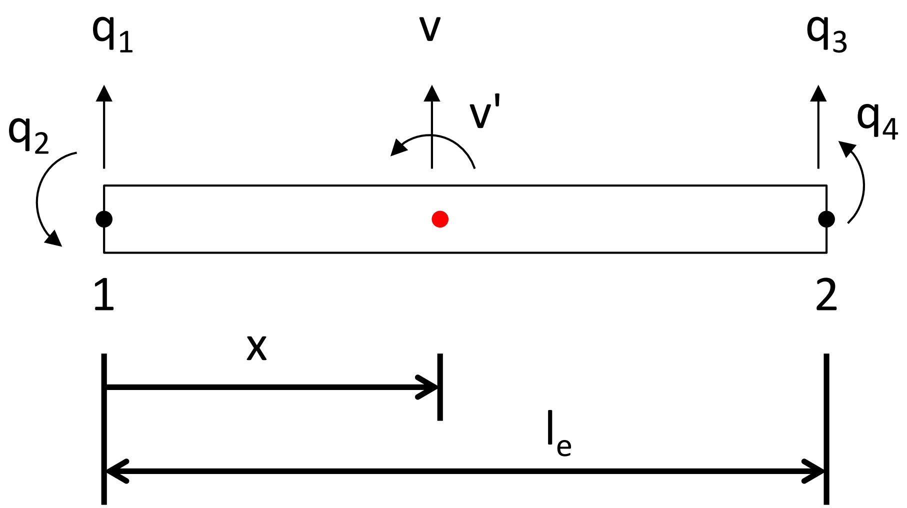




- Each node has 2 degrees of freedom
- One is the transverse displacement
- The other is the slope or rotation

Element Analysis (Beam element)

Beam Element



Note: All directions shown produces positive deformations

Natural Coordinate:

For a 2-noded beam element, we assume:

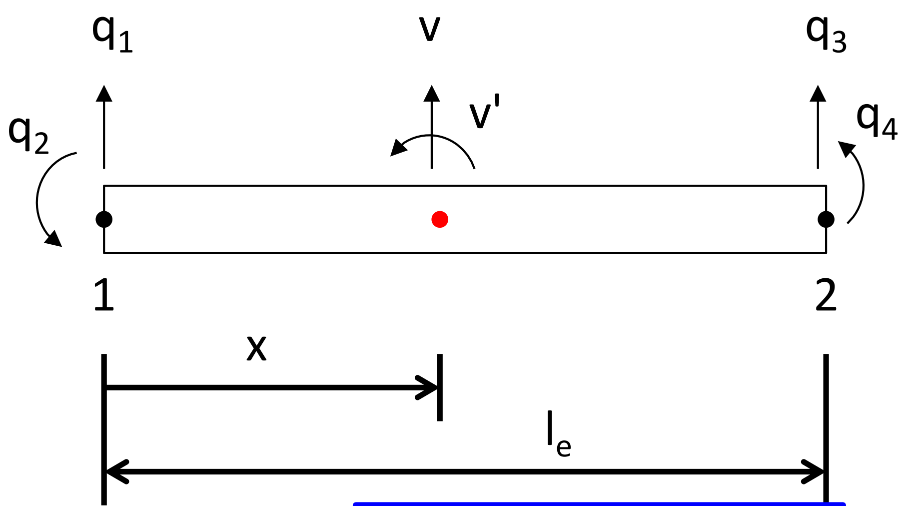
$$x_1 \rightarrow \xi = -1$$

$$x_2 \rightarrow \xi = +1$$

$$\boxed{\therefore \xi = 2\left(\frac{x}{l_e}\right) - 1}$$

Similar to 1-D bar element

Beam Element



$$H_1 = \left(\frac{\xi^3 - 3\xi + 2}{4} \right)$$

$$H_2 = \left(\frac{\xi^3 - \xi^2 - \xi + 1}{4} \right)$$

$$H_3 = \left(\frac{-\xi^3 + 3\xi + 2}{4} \right)$$

$$H_4 = \left(\frac{\xi^3 + \xi^2 - \xi - 1}{4} \right)$$

Also known as Hermite Shape Functions

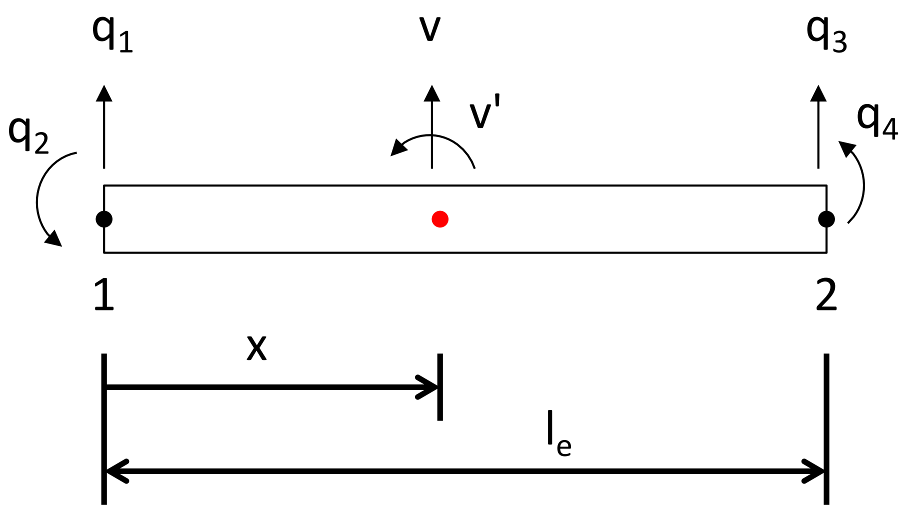
Shape Functions:

- Beam element has 4 degrees of freedom
- Appropriate to sue cubic function to describe displacement field

Displacement field:

$$v = H_1 q_1 + \frac{l_e}{2} H_2 q_2 + H_3 q_3 + \frac{l_e}{2} H_4 q_4$$

Beam Element



Shape Functions:

- Beam element has 4 degrees of freedom
- Appropriate to sue cubic function to describe displacement field

Slope:

$$v' = \frac{2}{l_e}\left(\frac{3\xi^2-3}{4}\right)q_1 + \left(\frac{3\xi^2-2\xi-1}{4}\right)q_2 + \frac{2}{l_e}\left(\frac{-3\xi^2+3}{4}\right)q_3 + \left(\frac{3\xi^2+2\xi-1}{4}\right)q_4$$

$$\boxed{v' = \left(\frac{2}{l_e}\left(\frac{3\xi^2-3}{4}\right) \quad \left(\frac{3\xi^2-2\xi-1}{4}\right) \quad \frac{2}{l_e}\left(\frac{-3\xi^2+3}{4}\right) \quad \left(\frac{3\xi^2+2\xi-1}{4}\right)\right)\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}}$$

Beam Element

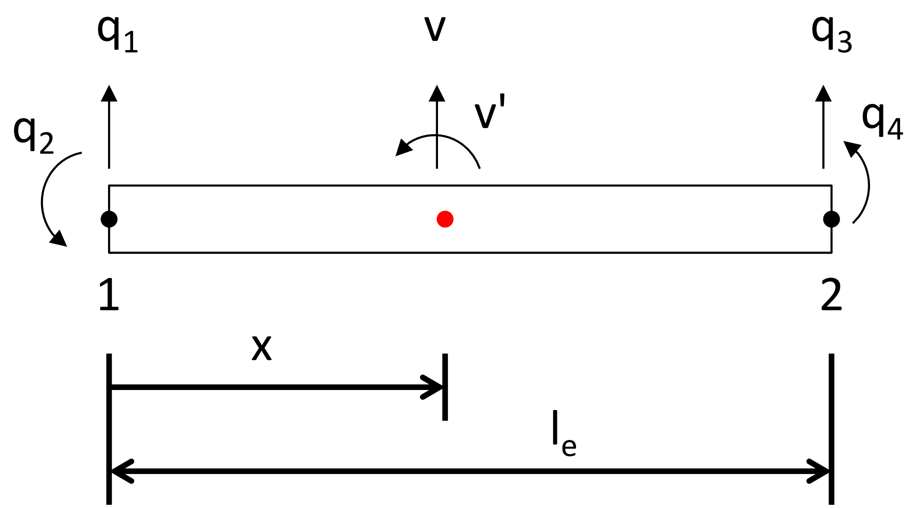
Potential Energy Approach: $$\Pi = U + \Omega$$

Strain Energy Work Potential

<u>For beam element</u>

$$\Pi = \frac{1}{2} EI \int_L \left(\frac{d^2 v}{dx^2} \right)^2 dx \boxed{- \int_L p v \, dx - \sum_m P_m v_m - \sum_k M_k v'_k}$$

Beam Element



Element Stiffness Matrix, k

For beam element, the element strain energy is given as:

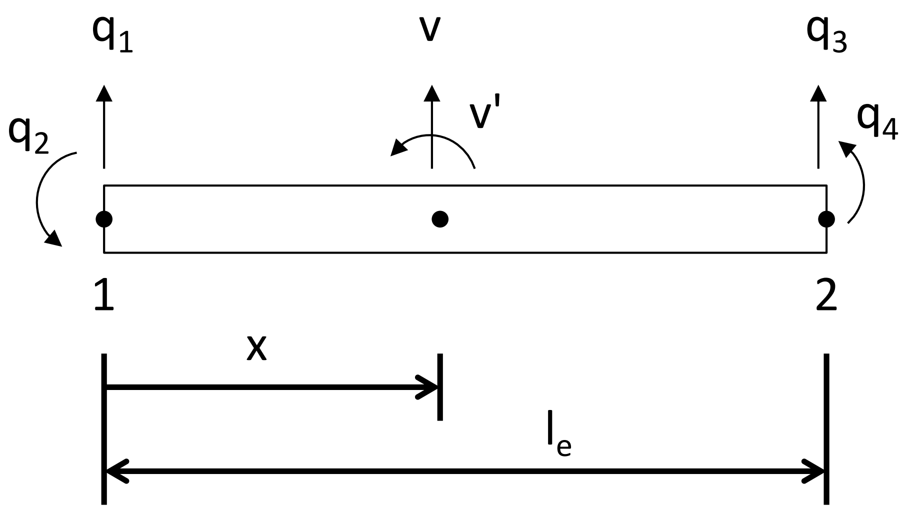
$$U_e = \frac{1}{2} EI \int \left(\frac{d^2 v}{dx_2} \right)^2 dx$$

$$\xi = 2\left(\frac{x}{l_e}\right) - 1$$

$$\frac{d^2 v}{dx_2} = \frac{d}{d\xi} \cdot \frac{dv}{dx} \cdot \frac{d\xi}{dx}$$

$$\therefore \frac{d\xi}{dx} = \frac{2}{l_e}$$

Beam Element



Element Stiffness Matrix, k

For beam element, the element strain energy is given as:

$$U_e = \frac{1}{2} EI \int \left(\frac{d^2 v}{dx_2} \right)^2 dx$$

$$\frac{d^2 v}{dx_2} = \frac{1}{l_e^2} \begin{pmatrix} 6\xi & l_e(3\xi - 1) & -6\xi & l_e(3\xi + 1) \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}$$

$$\therefore \frac{d^2 v}{dx_2} = \frac{1}{l_e^2} \{B\}\{q\}$$

Beam Element

$$U_e = \frac{1}{2}EI\int\left(\frac{1}{l_e^2}\{B\}\{q\}\right)^2 dx$$

$$U_e = \frac{1}{2}EI\int\left(\frac{1}{l_e^2}\{B\}\{q\}\right)^2 \frac{l_e}{2}d\xi$$

$$U_e = \frac{1}{2}EI\int\frac{1}{2l_e^3}\{q\}^T\{B\}^T\{B\}\{q\}d\xi$$

$$U_e = \frac{1}{2}\{q\}^T EI\int\frac{1}{2l_e^3}\{B\}^T\{B\}d\xi\{q\}$$

Element Stiffness Matrix, k

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$$U_e = \frac{1}{2}EI\int\left(\frac{d^2v}{dx_2}\right)^2 dx$$

$$\xi = 2\left(\frac{x}{l_e}\right) - 1$$

$$\frac{d\xi}{dx} = \frac{2}{l_e} \rightarrow \therefore dx = \frac{l_e}{2}d\xi$$

Beam Element

$$U_e = \frac{1}{2} EI \int \left(\frac{1}{l_e^2} \{B\}\{q\} \right)^2 dx$$

$$U_e = \frac{1}{2} EI \int \left(\frac{1}{l_e^2} \{B\}\{q\} \right)^2 \frac{l_e}{2} d\xi$$

$$U_e = \frac{1}{2} EI \int \frac{1}{2l_e^3} \{q\}^T \{B\}^T \{B\}\{q\} d\xi$$

$$U_e = \frac{1}{2} \{q\}^T EI \boxed{\int \frac{1}{2l_e^3} \{B\}^T \{B\} d\xi} \{q\}$$

Element Stiffness Matrix, k

For beam element, the element strain energy is given as:

$$U_e = \frac{1}{2} EI \int \left(\frac{d^2 v}{dx_2} \right)^2 dx$$

$$\boxed{\{B\}\{q\} = \{q\}^T \{B\}^T}$$

Beam Element

$$\int_{-1}^{1} \{B\}^T \{B\} d\xi$$

$$= \int_{-1}^{1} \begin{pmatrix} 6\xi \\ l_e(3\xi - 1) \\ -6\xi \\ l_e(3\xi + 1) \end{pmatrix} \begin{pmatrix} 6\xi & l_e(3\xi - 1) & -6\xi & l_e(3\xi + 1) \end{pmatrix} d\xi$$

$$= \int_{-1}^{1} \begin{pmatrix} 36\xi^2 & 6\xi(3\xi-1)l_e & -36\xi^2 & 6\xi(3\xi+1)l_e \\ 6\xi(3\xi-1)l_e & (3\xi-1)^2 l_e^2 & -6\xi(3\xi-1)l_e & (3\xi-1)(3\xi+1)l_e^2 \\ -36\xi^2 & -6\xi(3\xi-1)l_e & 36\xi^2 & -6\xi(3\xi+1)l_e \\ 6\xi(3\xi+1)l_e & (3\xi+1)(3\xi-1)l_e^2 & -6\xi(3\xi+1)l_e & (3\xi+1)^2 l_e^2 \end{pmatrix} d\xi$$

$$= \begin{pmatrix} 24 & 12l_e & -24 & 12l_e \\ 12l_e & 8l_e^2 & -12l_e & 4l_e^2 \\ -24 & -12l_e & 24 & -12l_e \\ 12l_e & 4l_e^2 & -12l_e & 8l_e^2 \end{pmatrix}$$

Beam Element

Element Stiffness Matrix, k

For beam element, the element strain energy is given as:

$$U_e = \frac{1}{2} EI \int \left(\frac{d^2 v}{dx_2} \right)^2 dx$$

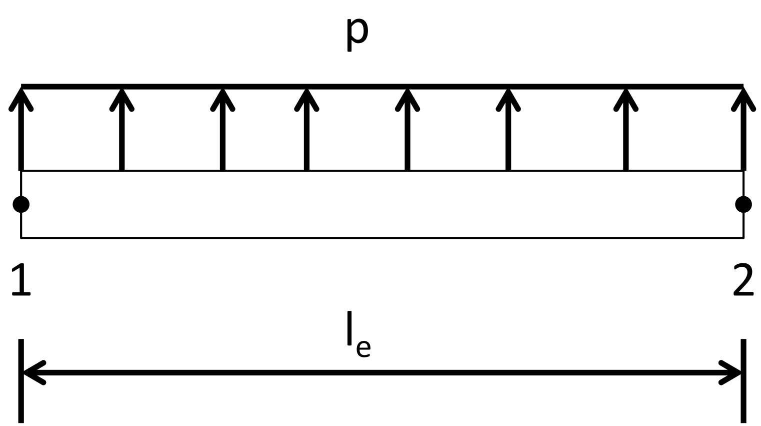
$$U_e = \frac{1}{2}\{q\}^T \frac{EI}{2l_e^3} \begin{pmatrix} 24 & 12l_e & -24 & 12l_e \\ 12l_e & 8l_e^2 & -12l_e & 4l_e^2 \\ -24 & -12l_e & 24 & -12l_e \\ 12l_e & 4l_e^2 & -12l_e & 8l_e^2 \end{pmatrix} \{q\}$$

$$\therefore U_e = \frac{1}{2}\{q\}^T k_e \{q\}$$

With

$$\boxed{k_e = \frac{EI}{l_e^3} \begin{pmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{pmatrix}}$$

Beam Element



$$\int_L pv\,dx$$

$$= \int_{-1}^{1} p \begin{pmatrix} H_1 & \frac{l_e}{2}H_2 & H_3 & \frac{l_e}{2}H_4 \end{pmatrix} \{q\} \frac{l_e}{2} d\xi$$

$$= \frac{pl_e}{2} \boxed{\int_{-1}^{1} \begin{pmatrix} H_1 & \frac{l_e}{2}H_2 & H_3 & \frac{l_e}{2}H_4 \end{pmatrix} d\xi} \{q\}$$

Load vector: Distributed load, p

$$\int_L pv\,dx$$

$$\xi = 2\left(\frac{x}{l_e}\right) - 1$$

$$\frac{d\xi}{dx} = \frac{2}{l_e} \rightarrow \therefore dx = \frac{l_e}{2} d\xi$$

$$v = \begin{pmatrix} H_1 & \frac{l_e}{2}H_2 & H_3 & \frac{l_e}{2}H_4 \end{pmatrix} \{q\}$$

Beam Element

$$H_1 = \left(\frac{\xi^3 - 3\xi + 2}{4} \right)$$

$$H_2 = \left(\frac{\xi^3 - \xi^2 - \xi + 1}{4} \right)$$

$$H_3 = \left(\frac{-\xi^3 + 3\xi + 2}{4} \right)$$

$$H_4 = \left(\frac{\xi^3 + \xi^2 - \xi - 1}{4} \right)$$

$$\Rightarrow$$

$$\int_{-1}^{1} H_1\, d\xi = \int_{-1}^{1} \left(\frac{\xi^3 - 3\xi + 2}{4} \right) d\xi = 1$$

$$\int_{-1}^{1} H_2\, d\xi = \int_{-1}^{1} \left(\frac{\xi^3 - \xi^2 - \xi + 1}{4} \right) d\xi = \frac{1}{3}$$

$$\int_{-1}^{1} H_3\, d\xi = \int_{-1}^{1} \left(\frac{-\xi^3 + 3\xi + 2}{4} \right) d\xi = 1$$

$$\int_{-1}^{1} H_4\, d\xi = \int_{-1}^{1} \left(\frac{\xi^3 + \xi^2 - \xi - 1}{4} \right) d\xi = -\frac{1}{3}$$

Beam Element

$$\int_L pv\,dx$$

$$= \int_{-1}^{1} p\begin{pmatrix} H_1 & \frac{l_e}{2}H_2 & H_3 & \frac{l_e}{2}H_4 \end{pmatrix}\{q\}\frac{l_e}{2}d\xi$$

$$= \frac{pl_e}{2}\begin{pmatrix} 1 & \frac{l_e}{6} & 1 & -\frac{l_e}{6} \end{pmatrix}\{q\}$$

$$= \begin{pmatrix} \frac{pl_e}{2} & \frac{pl_e^2}{12} & \frac{pl_e}{2} & -\frac{pl_e^2}{12} \end{pmatrix}\{q\}$$

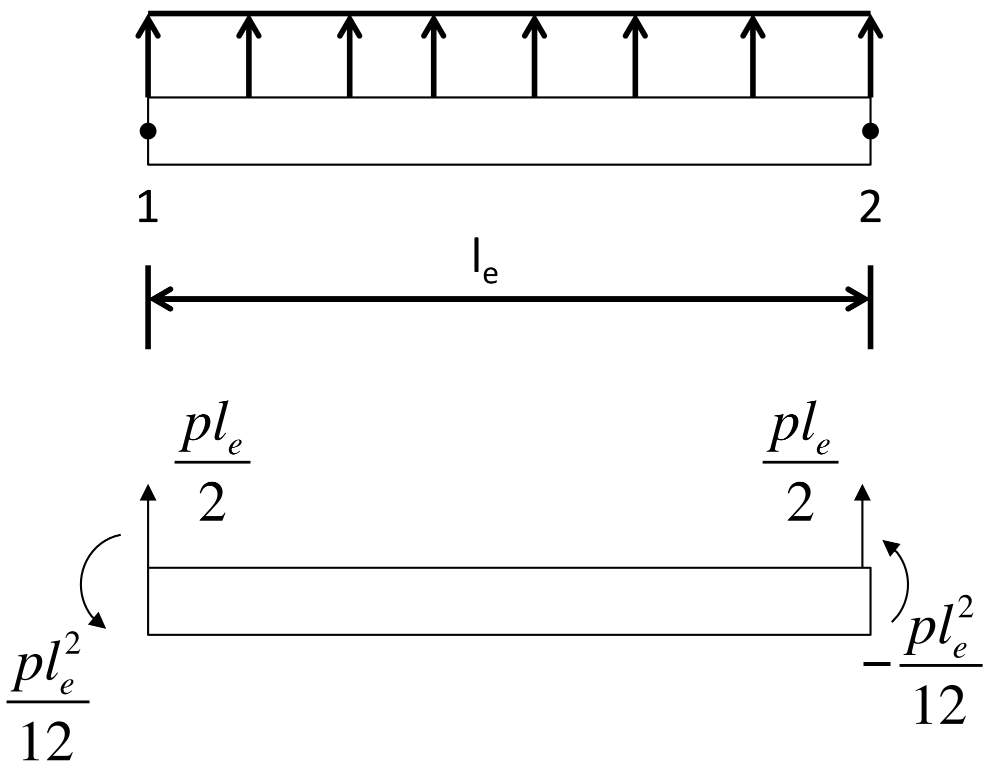
$$= \{f_e\}\{q\}$$

Load vector: Distributed load, p

$$\int_L pv\,dx$$

$$\boxed{\{f_e\} = \begin{pmatrix} \frac{pl_e}{2} & \frac{pl_e^2}{12} & \frac{pl_e}{2} & -\frac{pl_e^2}{12} \end{pmatrix}}$$

Beam Element



Load vector: Distributed load, p

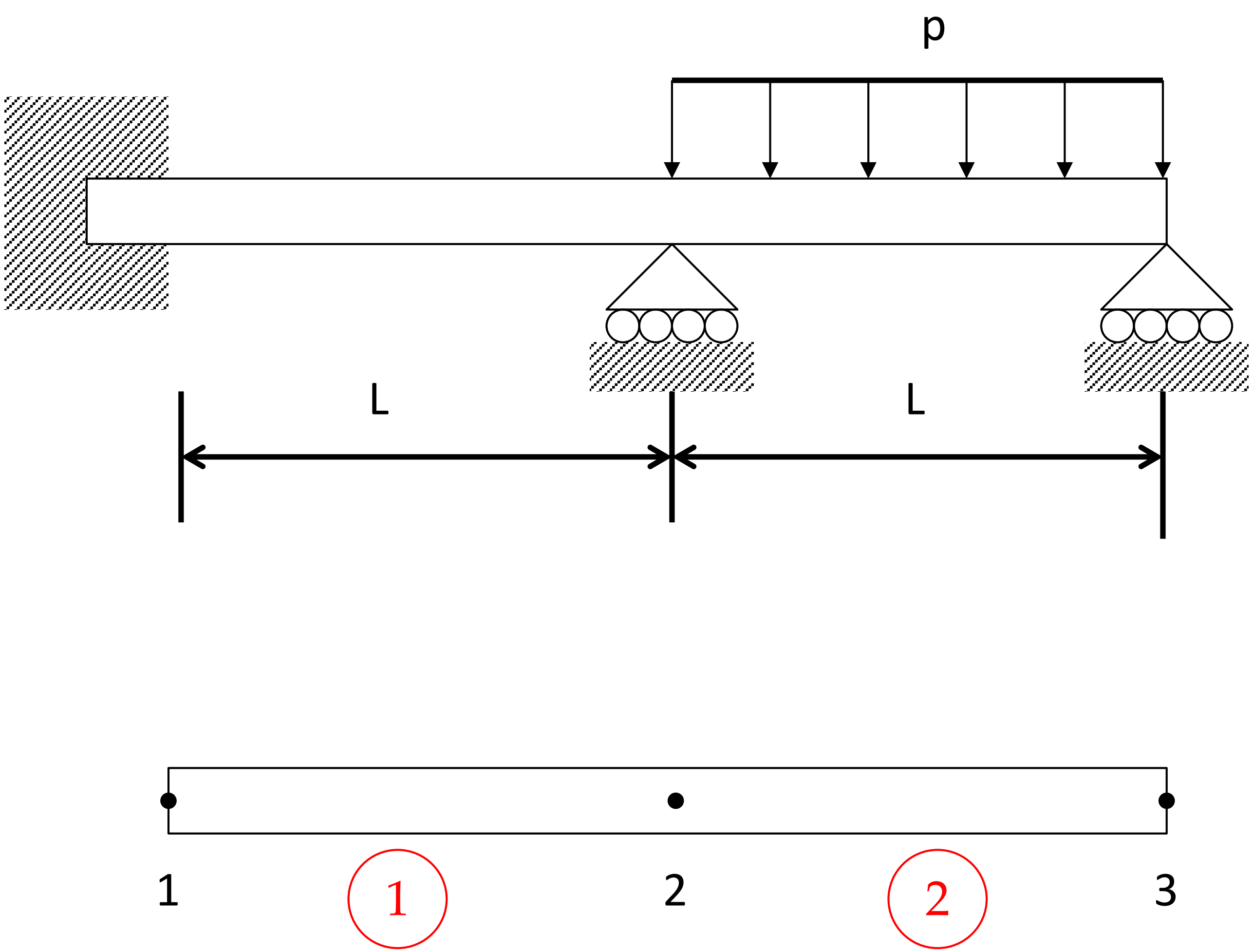
$$\int_L pv\,dx$$

q1 q2 q3 q4

$$\boxed{\{f_e\} = \begin{pmatrix} \frac{pl_e}{2} & \frac{pl_e^2}{12} & \frac{pl_e}{2} & -\frac{pl_e^2}{12} \end{pmatrix}}$$

Assembly of Global Stiffness Matrix

Global Stiffness Matrix

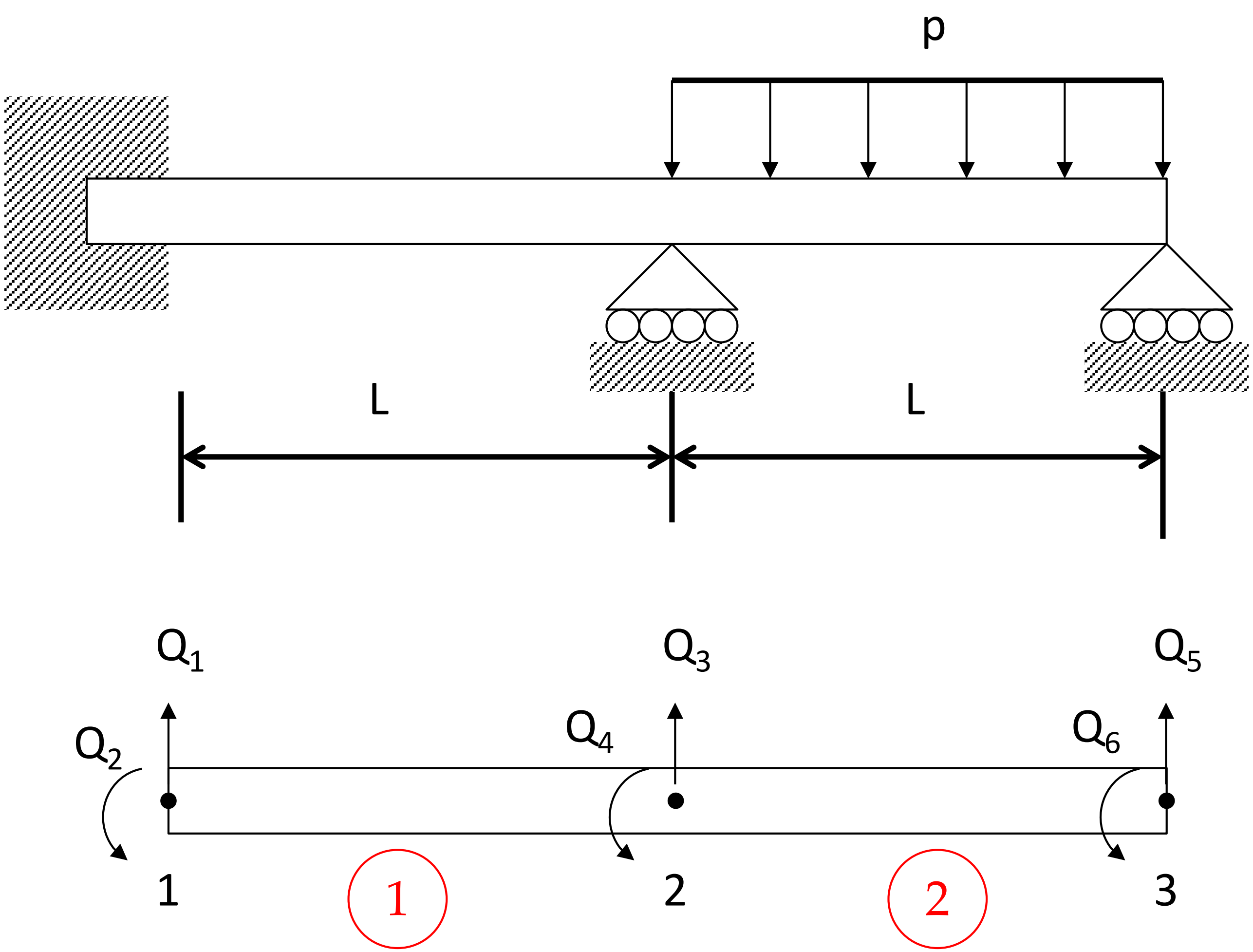


Assemble the global stiffness for the beam structure given using 2 beam elements.

Element Connectivity Table

Element	Nodes	
1	1	2
2	2	3

Global Stiffness Matrix



Assemble the global stiffness for the beam structure given using 2 beam elements.

Element Connectivity Table

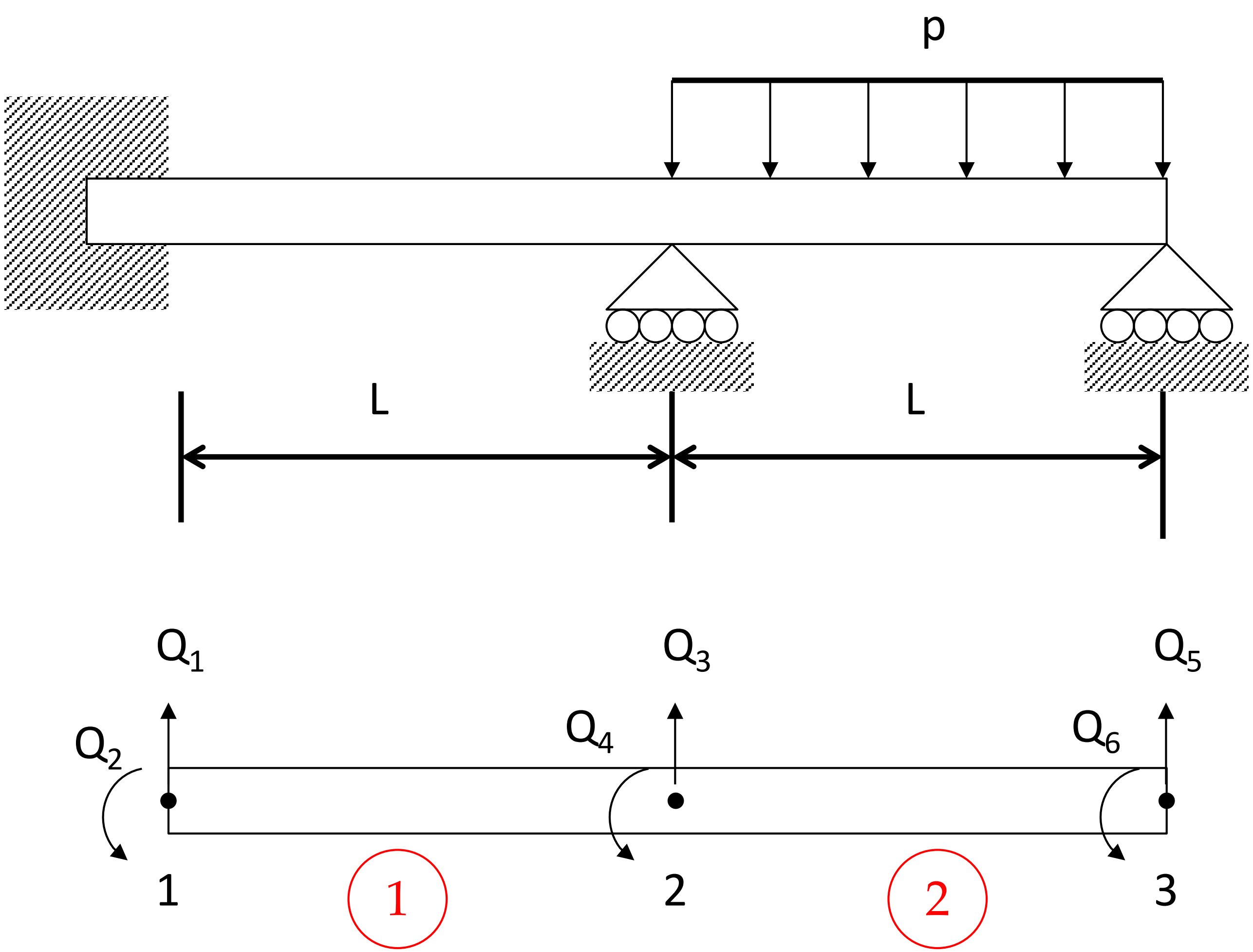
Element	Nodes	
1	1	2
2	2	3

Global Stiffness Matrix

- Problem consists of 3 nodes, each with 2 degrees of freedom
- Therefore, stiffness matrix size is 6X6

$$
K = \frac{EI}{L^3}
\begin{array}{c c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} & \\
\begin{pmatrix}
k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\
k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\
k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} \\
k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} \\
k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} \\
k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66}
\end{pmatrix} &
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\end{array}
$$

Global Stiffness Matrix



Assemble the global stiffness for the beam structure given using 2 beam elements.

Given:

$$k_e = \frac{EI}{l_e^3}\begin{pmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{pmatrix}$$

Global Stiffness Matrix

$$k_1 = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{pmatrix} \end{array}\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

$$K = \frac{EI}{L^3}\begin{pmatrix} 12 & 6L & -12 & 6L & k_{15} & k_{16} \\ 6L & 4L^2 & -6L & 2L^2 & k_{25} & k_{26} \\ -12 & -6L & 12 & -6L & k_{35} & k_{36} \\ 6L & 2L^2 & -6L & 4L^2 & k_{45} & k_{46} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} \end{pmatrix}$$

$$k_e = \frac{EI}{l_e^3}\begin{pmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{pmatrix}$$

Global Stiffness Matrix

$$k_1 = \frac{EI}{L^3}\begin{array}{cc} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\ \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$

$+$

$$k_2 = \frac{EI}{L^3}\begin{array}{cc} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} & \\ \begin{pmatrix} k_{33} & k_{34} & k_{35} & k_{36} \\ k_{43} & k_{44} & k_{45} & k_{46} \\ k_{53} & k_{54} & k_{55} & k_{56} \\ k_{63} & k_{64} & k_{65} & k_{66} \end{pmatrix} & \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

$$K = \frac{EI}{L^3}\begin{pmatrix} 12 & 6L & -12 & 6L & k_{15} & k_{16} \\ 6L & 4L^2 & -6L & 2L^2 & k_{25} & k_{26} \\ -12 & -6L & 12+12 & -6L+6L & -12 & 6L \\ 6L & 2L^2 & -6L+6L & 4L^2+4L^2 & k_{45} & 2L \\ k_{51} & k_{52} & -12 & -6L & 12 & -6L \\ k_{61} & k_{62} & 6L & 2L^2 & -6L & 4L^2 \end{pmatrix}$$

$$\boxed{k_e = \frac{EI}{l_e^3}\begin{pmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{pmatrix}}$$

Global Stiffness Matrix

$$
\begin{array}{cccccccc}
 & & 1 & 2 & 3 & 4 & 5 & 6 \\
K = \dfrac{EI}{L^3} & \left(\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\right. & \begin{array}{c} 12 \\ 6L \\ -12 \\ 6L \\ 0 \\ 0 \end{array} & \begin{array}{c} 6L \\ 4L^2 \\ -6L \\ 2L^2 \\ 0 \\ 0 \end{array} & \begin{array}{c} -12 \\ -6L \\ 24 \\ 0 \\ -12 \\ 6L \end{array} & \begin{array}{c} 6L \\ 2L^2 \\ 0 \\ 8L^2 \\ -6L \\ 2L^2 \end{array} & \begin{array}{c} 0 \\ 0 \\ -12 \\ -6L \\ 12 \\ -6L \end{array} & \begin{array}{c} 0 \\ 0 \\ 6L \\ 2L \\ -6L \\ 4L^2 \end{array} \left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\right) \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\end{array}
$$

By the end of the notes:

You are expected to be able:

- To assemble the stiffness matrix and load vector for a beam element
- To assemble the global stiffness matrix for a beam structure